1 Premilinaries

Fuzzy set theory and fuzzy logic is viewed as a generalisation of ordinary set theory and logic. To this vein, we will first explore set theory and logic in a slightly rigourous fashion so that when we consider its generalisation, we do not end up feeling lost in a technical jargon but are, rather, able to relate directly to our previous concepts.

1.1 Propositional Logic

Propositional Logic of first order PL(1) consists of syntax (grammar), semantics (meaning), inference rules and derivation. A rule of inference, inference rule, or transformation rule is a logical form consisting of a function which takes premises, analyzes their syntax, and returns a conclusion (or conclusions). For example, the rule of inference called *modus ponens* takes two premises, one in the form "If p then q" and another in the form "p", and returns the conclusion "q". A derivation, on the other hand, is the conclusion of the argument via inference.

PL can be viewed as a language of human reasoning and this language is based on alphabets i.e. symbols

The alphabets or primitive symbols of PL consist of

a) Propositional variables denoted by $p, q, r, s, t, ...$

b) Constants denoted by T and C

c) Connectives denoted by $\vee, \wedge, N\ \rightarrow, \leftrightarrow$, respectively called disjunction, conjunction, negation, conditional and biconditional

Just as in any language, syntax or grammar is used to generate sentences. In PL too, syntax is used to generate well-formed formulae (WFFs) which are analogous to sentences. The WFFs are characterised recursively or inductively as follows:

a) All propositional variables and constants are WFFs (called primitive WFFs)

b) The negation of a WFF is a WFF

c) Disjunction, conjunction, conditional and biconditional of a pair of WFFs is also a WFF

d) All WFFs are obtained by the above three procedures applied for a finite number of times.

Remark 1 *The connective N will be used as prefix before a propositional symbol or WFF*

Remark 2 *All other connectives will be used as infix between a pair of WFFs or symbols*

Remark 3 *For clarity of understanding, we use certain extra symbols that are not part of the alphabets of PL(1). These symbols will be called meta-symbols. Some of them are brackets.*

Example 4 $p, p \vee q, p \wedge q, Np, p \rightarrow q, p \leftrightarrow q$ *are WFFs*

Example 5 $p\vee, pN, pq, pq \rightarrow, \wedge p$ *are not WFFs*

1.1.1 Semantics of PL(1)

Just as there is a dictionary for words, phrases and sentences of a natural language, giving their meaning, analogously, we talk of semantics of WFFs in PL(1). The dictionary of PL(1) is concise and compact as every primitive WFF can have only one of the two meanings – true or false. Given a WFF F, an interpretation of F is the assignment of one of the two values – true or false – to each propositional symbol, occuring in F. More generally, given a finite set S of WFFs, an interpretation of S is the assignment of one of the two values – true or false – to each propositional symbol occurring in each WFF F in S.

Remark 6 *For a single propositional symbol* p*, there are only 2 interpretations,* T *and* F*.*

Remark 7 *For a pair of propositional symbols, there are exactly 4 interpretations. In general, for a WFF with* n *propositional symbols, there are* 2^n *possible interpretations.*

Remark 8 *The meaning of T (tautology) and C (contradiction) are fixed. T is always true and C is always false.*

Definition 9 *A valuation of a WFF F associated with an interpretation, called the* ***meaning*** *of* F*, is the truth value of* F

Example 10 *Consider the WFF* $F := [(p \wedge q) \rightarrow r] \rightarrow [p \wedge (q \rightarrow r)]$

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$	$(q \rightarrow r)$	$p \wedge (q \rightarrow r)$	F
1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	0	1	1	0	0
0	1	0	0	1	0	0	0
0	0	1	0	1	1	0	0
0	0	0	0	1	1	0	0

Remark 11 *A WFF is said to be valid in an interpretation I if it is true in I*

Remark 12 *A finite set S of WFFs is said to be valid in an interpretation I if each WFF of S is valid in I*

Remark 13 *A WFF F is said to be valid if it is valid in every possible interpretation of F*

Remark 14 *A finite set S of WFFs is said to be valid if every WFF of S is valid in every possible interpretation of F*

Remark 15 *Two WFFs, F and G, are said to be equivalent, written $F \equiv G$, if the WFF $F \leftrightarrow G$ is valid.*

Example 16 $p \rightarrow q \equiv Np \vee q$

p	q	Np	$p \rightarrow q$	$Np \vee q$	$(p \rightarrow q) \leftrightarrow (Np \vee q)$
1	1	0	1	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

Exercise 17 *Show that* $N(p \rightarrow q) \equiv p \wedge Nq$

Solution 18

p	q	$p \rightarrow q$	$N(p \rightarrow q)$	Nq	$p \wedge Nq$	$N(p \rightarrow q) \leftrightarrow p \wedge Nq$
1	1	1	0	0	0	1
1	0	0	1	1	1	1
0	1	1	0	0	0	1
0	0	1	0	1	0	1

Exercise 19 *Show that* $p \leftrightarrow q \equiv (Np \vee q) \vee (p \vee Nq) \equiv (p \wedge q) \vee [(Np) \wedge (Nq)]$

Solution 20 *Let* $(p \wedge q) \vee [(Np) \wedge (Nq)] = G$ *and* $(Np \vee q) \vee (p \vee Nq) = F$

p	q	$p \leftrightarrow q$	Np	$Np \vee q$	Nq	$p \vee Nq$	F	$(p \leftrightarrow q) \leftrightarrow F$
1	1	1	0	1	0	1	1	1
1	0	0	0	0	1	1	0	1
0	1	0	1	1	0	0	0	1
0	0	1	1	1	1	1	1	1

p	q	Np	Nq	$p \wedge q$	$(Np) \wedge (Nq)$	G	$p \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow G$
1	1	0	0	1	0	1	1	1
1	0	0	1	0	0	0	0	1
0	1	1	0	0	0	0	0	1
0	0	1	1	0	1	1	1	1

1.2 Predicate Logic

Predicate logic PL(2) is a natural extension of propositional logic PL(1). PL(1) deals with WFFs that do not involve variables whereas PL(2) deals with WFFs involving variables

1.2.1 Syntax of PL(2)

The syntax of PL(2) consists of the symbol set of alphabets and rules. The alphabets of PL(2) consist of

a) Constants $a, b, c, ...$
b) Variables $x, y, z, ...$
c) Truth symbols T and C
d) Predicate symbols $P, Q, R, ..$
e) Function symbols $f, g, h, ...$

f) Connectives $\vee, \wedge, N \rightarrow, \leftrightarrow$

g) Quantifiers: the inverted E, $\exists$, called the existential quantifier, pronounced as "there exist", "for some", "at least one" and the inverted A, $\forall$, called the universal quantifier, pronounced as "for all", "for every" and "for each".

Notice that in this case, we have only two truth valuations. That is, $I_2 = \{0, 1\}$. Now, we start with $\mathcal{U}$ having values in $I_3 = \{0, 1/2, 1\}$. This is 3-valued logic. This leads to a new kind of set theory, namely the 3-valued set-theory but will be of little interest to us in our fuzzy considerations. For now, here are some details:

Let $F(\mathcal{U})$ be the set of all functions from $\mathcal{U}$ to I_3. We want to define 3 operations: union, intersection and complementation on $F(\mathcal{U})$

$$\begin{array}{cccc} u & 0 & 1/2 & 1 \\ 0 & 0 & 1/2 & 1 \\ 1/2 & 1/2 & 1/2 & 1 \\ 1 & 1 & 1 & 1 \end{array} \quad \text{and} \quad \begin{array}{cccc} m & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 1/2 \\ 1 & 0 & 1/2 & 1 \end{array}$$

where $u = \max(a, b)$, $m = \min(a, b)$ and $c(a) = 1 - a$

1.3 Basic Set Theory

This part will assume familiarity with functions, sets and set operations in general including the Cartesian product, union and intersection and its variants and (binary) relations. Generalisation of the union and intersections are studied in the later part of this chapter whereas a rigorour introduction to Order Theory and Lattice Theory is offered in the next.

Definition 21 *Let $\mathcal{U}$ be a fixed non-empty universal set. The function $f : \mathcal{U} \longrightarrow \{0, 1\}$ is called a **characteristic function** or **indicator function** of $\mathcal{U}$.*

Given any characteristic function f, we can associate a unique subset A of $\mathcal{U}$, namely $A_f = \{x \in \mathcal{U} : f(x) = 1\}$

Conversely, given any subset A of $\mathcal{U}$, we can associate a unique characterisitic function f on $\mathcal{U}$ namely

$$f_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

This acts as the Boolean operator "belongs to is true" and "belongs to is false".

Theorem 22 *If A and B are subsets of $\mathcal{U}$, then*

1. $f_{A \cup B} = \max(f_A, f_B)$
2. $f_{A \cap B} = \min(f_A, f_B)$
3. $f_{A^c} = 1 - f_A$

Proof. $f_{A\cup B}(x) = \begin{cases} 1 \text{ if } x \in A \text{ or } B \\ 0 \text{ if } x \notin \text{either } A \text{ or } B \end{cases}$

Consider the following cases:

1. $f_A(x) = 1$ and $f_B(x) = 0$, then, $f_{A\cup B}(x) = 1$
2. $f_A(x) = 0$ and $f_B(x) = 1$, then, $f_{A\cup B}(x) = 1$
3. $f_A(x) = 0$ and $f_B(x) = 0$, then, $f_{A\cup B}(x) = 0$
4. $f_A(x) = 1$ and $f_B(x) = 1$, then, $f_{A\cup B}(x) = 1$

In all such cases, the definition $\max(f_A, f_B)$ coincides with $f_{A\cup B}$

The proof of part 2 is similar

For part three, consider only the two cases for $f_A(x) = 1$ and 0 ■

Exercise 23 *Let f and g be characteristic functions on $\mathcal{U}$. Define the binary operation $\to$ by*

$$f \to g = \begin{cases} 0 \textit{ if } f = 1 \textit{ and } g = 0 \\ 1 \textit{ otherwise} \end{cases}$$

a) Write down the table for $\to$

b) Prove that $f \to g$ is a characteristic function on $\mathcal{U}$

c) Prove that $f \to g = \max(Nf, g)$ where $Nf = \begin{cases} 0 \textit{ if } f = 1 \\ 1 \textit{ otherwise} \end{cases}$

d) If $A = \{x \in \mathcal{U} \mid f(x) = 1\}$ and $B = \{x \in \mathcal{U} \mid g(x) = 1\}$, prove that "$f \to g = 1$" if and only if $A \subseteq B$

Solution 24 *a)*

f	g	$f \to g$
1	1	1
1	0	0
0	1	1
0	0	1

b) We have

$$(f \to g)(x) = \begin{cases} 0 \textit{ if } f(x) = 1 \textit{ and } g(x) = 0 \\ 1 \textit{ otherwise} \end{cases}$$

The domain of $(f \to g)(x)$ relies on the domain of both f and g, which is $\mathcal{U}$. The range is $\{0, 1\}$

c) This can be done by considering every single case for f and g.

d) ($\Longrightarrow$) Let $f \to g = 1$. We will have three different cases.

Case I

$f(x) = 1$ and $g(x) = 1$

We can rephase this as "if f(x) = 1, then $g(x) = 1$" which gives us $A \subseteq B$

Case II

$f(x) = 0$ and $g(x) = 1$

If $f(x) = 0$, then we have the empty set since $f(x) = 0$ for any $x \in \mathcal{U}$. Since the empty set is trivially the subset of everyset, therefore $A \subseteq B$

Case III

$f(x) = 0$ and $g(x) = 0$.

If $g(x) = 0$, then $f(x) = 0$. That is, if $x \notin B$, then $x \notin A$. Hence, $B^c \subseteq A^c \iff A \subseteq B$

($\Longleftarrow$) *If* $A \subseteq B$, *then for* $x \in A$, *we have* $x \in B$. *Hence,* $f(x) = 1$ *implies* $g(x) = 1$. *Thus,* $(f \rightarrow g)(x) = 1$. *Since this is valid for any* x, *we have* $f \rightarrow g = 1$

If S and R are binary relations from A to B and from B to C, respectively. Show that the function $f_{S \circ R}(a, c) = \max\{\min\{f_S(a, b), f_R(b, c)\} \, b \in B\}$. To prove that $f_{S \circ R}$ is indeed a characterisitic function, one needs to show that the range of $f_{S \circ R}$ is $\{0, 1\}$ and that the domain of $f_{S \circ R}$ is $\mathcal{U} \times \mathcal{U}$. The set $S \circ R := \{(a, c) \mid \exists b \text{ s.t } (a, b) \in S \text{ and } (b, c) \in R\}$. This makes sense because if $f_S(a, b) = 1$, then $(a, b) \in S$ and $f_R(b, c) = 1$, then $(b, c) \in R$. Hence, we can collect all such a's and c's and form the set $\{(a, c) \mid a \in A \text{ and } c \in C\}$ with a characteristic function called $f_{S \circ R}$. In the definition, even if we have one $b \in B$ such that $(a, b) \in S$ and $(b, c) \in R$, then the maximum value of all (a, b) and (b, c) for varying b will be 1. If no such b is found, then the value will be 0. The minimum function guarantees that both $f_S(a, b)$ and $f_R(b, c) = 1$, that is, we do have $(a, b) \in S$ and $(b, c) \in R$ to begin with. The range is then clearly $\{0, 1\}$ and the domain clearly the cartesian product $\mathcal{U} \times \mathcal{U}$.

Before we move any further, a definition is in order. $\vee$ is a binary function such that $\vee(x, y) = x \vee y = \sup\{x, y\}$ and $\wedge(x, y) = x \wedge y = \inf\{x, y\}$. More details about this under the discussion of lattices.

Proposition 25 *Let* $I_2 = \{0, 1\}$ *and* $Ch(\mathcal{U})$ *be the set of characteristic functions on universal set* $\mathcal{U}$. *Then,* $f, g \in Ch(U) \implies f \vee g,\ f \wedge g, Nf \in Ch(U)$

Proof. Define $(f \vee g)(x) = f(x) \vee g(x)$, $(f \wedge g)(x) = f(x) \wedge g(x)$ and

$$Nf(x) = \begin{cases} 1 & \text{if } f(x) = 0 \\ 0 & \text{if } f(x) = 1 \end{cases}$$

If $A = \{x \mid f(x) = 1\}$ and $B = \{x \mid g(x) = 1\}$, then $f \vee g$, $f \wedge g$, Nf construct the sets $A \cup B$, $A \cap B$ and A^c, respectively so that they do, indeed, form charactersitic functions. ■

Note that $f(x) \vee g(x)$ can be defined in a multitude of ways. For instance, $\max(f(x), g(x))$, $f(x) + g(x) - f(x)g(x)$. Similarly, $(f \wedge g)(x)$ might correspond to $\min(f(x), g(x))$, $f(x)g(x)$

Definition 26 *Let* $I_2 = \{0, 1\}$. *Then,* $u : I_2 \times I_2 \longrightarrow I_2$, $m : I_2 \times I_2 \longrightarrow I_2$ *and* $c : I_2 \longrightarrow I_2$ *such that* $u(a, b) = a + b - ab$ *and* $m(a, b) = ab$ *and* $c(a) = 1 - a$ *are called the* ***union, meet*** *and* ***complement operators.***

Proposition 27 $u(a, a) = a$

Proof. Proof by exhaustion
$u(1, 1) = 1 + 1 - 1 = 1$
$u(0, 0) = 0 + 0 - 0 = 0$ ■

Proposition 28 $m(a, a) = a$

Proof. Proof by exhaustion

$m(1,1) = (1)(1) = 1$

$m(0,0) = (0)(0) = 0$ ■

Proposition 29 $u(a,b) = u(b,a)$

Proof. $u(a,b) = a + b - ab = b + a - ba = u(a,b)$ ■

Proposition 30 $m(a,b) = m(b,a)$

Proof. $m(a,b) = ab = ba = m(a,b)$ ■

Proposition 31 $u(a, u(b,c)) = u(u(a,b), c)$

Proof. $u(a, u(b,c)) = a + u(b,c) - au(b,c) =$

$= a + (b + c - bc) - a(b + c - bc)$

$= a + b + c - bc - ab - ac + abc$

$= (a + b) - ab + c - c(a + b - ab)$

$= u(a,b) + c - cu(a,b)$

$= u(u(a,b), c)$ ■

Proposition 32 $m(a, m(b,c)) = m(m(a,b), c)$

Proof. $m(a, m(b,c)) = a(bc) = (ab)c = m(m(a,b), c)$ ■

Proposition 33 $u(a, m(b,c)) = m(u(a,b), u(a,c))$

Proof. $u(a, m(b,c)) = u(a, bc)$

$= a + bc - abc$

$= a^2 + bc - abc - ab + ab + ac - ac + abc - abc$

$= a^2 + ac - a^2c + ba + bc - abc - a^2b - abc + a^2bc$

$= (a + b - ab)(a + c - ac)$

$= u(a,b)\, u(a,c)$

$= m(u(a,b), u(a,c))$ ■

Proposition 34 $m(a, u(b,c)) = u(m(a,b), m(a,c))$

Proof. $m(a, u(b,c)) = au(b,c)$

$= a(b + c - bc) = ab + ac - abc$

$= ab + ac - a^2bc$

$= ab + ac - abac$

$= m(a,b) + m(a,c) - m(a,b)\, m(a,c)$

$= u(m(a,b), m(a,c))$ ■

Proposition 35 $u(a, m(a,b)) = a$

Proof. $u(a, m(a, b)) = a + m(a, b) - am(a, b)$
$= a + ab - aab$
$= a + ab - ab = a$ ■

Proposition 36 $m(a, u(a, b)) = a$

Proof. $m(a, u(a, b)) = au(a, b)$
$= a(a + b - ab)$
$= a^2 + ab - a^2b$
$= a + ab - ab = a$ ■

Proposition 37 $u(a, 1) = 1$

Proof. $a + 1 - a1$
$= a + 1 - a = 1$ ■

Proposition 38 $u(a, 0) = a$

Proof. $a + 0 - a0 = a$ ■

Proposition 39 $m(a, 1) = a$

Proof. $a1 = a$ ■

Proposition 40 $m(a, 0) = 0$

Proof. $a0 = 0$ ■

Proposition 41 $c(c(a)) = a$

Proof. $1 - (1 - a)$
$= 1 - 1 + a = a$ ■

Proposition 42 $c(0) = 1$

Proof. $c(0) = 1 - 0 = 1$ ■

Proposition 43 $c(1) = 0$

Proof. $c(1) = 1 - 1 = 0$ ■

Proposition 44 $c(u(a, b)) = m(c(a), c(b))$

Proof. $c(u(a, b)) = 1 - u(a, b)$
$= 1 - a - b + ab$
$= (1 - a) - b(1 - a)$
$= (1 - a)(1 - b) = m(c(a), c(b))$ ■

Proposition 45 $c(m(a, b)) = u(c(a), c(b))$

Proof. $c\,(m\,(a,b)) = 1 - ab$
$= 1 + 1 - 1 - a - b + a + b - ab$
$= (1-a) + (1-b) - 1 + b + a - ab$
$= (1-a) + (1-b) - (1-a)\,(1-b)$
$= u\,(c\,(a)\,, c\,(b))$ ■

Proposition 46 $u\,(a, c\,(a)) = 1$

Proof. $a + 1 - a - (1-a)\,(a)$
$= 1 - a + a^2$
$= 1 - a + a = 1$ ■

Proposition 47 $m\,(a, c\,(a)) = 0$

Proof. $a\,(1-a) = a - a^2 = a - a = 0$ ■

We can generalise m and u a little further. A corresponding generalisation of u is as follows:

Definition 48 *A binary operation* $\triangle : \{0,1\} \times \{0,1\} \longrightarrow \{0,1\}$ *is a* ***t-norm*** *if it satisfies the following:*
a) $1 \triangle x = x$
b) $x \triangle y = y \triangle x$
c) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$
d) $w \leq x$ *and* $y \leq z$ *implies* $w \triangle z \leq x \triangle y$

Proposition 49 $0 \triangle x = 0$

Proof. Trivially, $0 \triangle x \geq 0$. Since $0 \leq x$ and $0 \leq 1$, then $0 \triangle x \leq 0 \triangle 1 = 0$. That is, $0 \triangle x \leq 0$. Combining the two inequalities, the proof is established. ■

Example 50 $x \triangle_0 y = \begin{cases} x \wedge y & \textit{if } x \vee y = 1 \\ 0 & \textit{otherwise} \end{cases}$

Example 51 $x \triangle_1 y = 0 \vee (x + y - 1)$

Example 52 $x \triangle_2 y = \frac{xy}{2-(x+y-xy)}$

Example 53 $x \triangle_3 y = xy$

Example 54 $x \triangle_4 y = \frac{xy}{x+y-xy}$

Example 55 $x \triangle_5 y = x \wedge y$

Proposition 56 *For any t-norm* $\triangle$*,* $\triangle_0 \leq \triangle \leq \triangle_5$

Proof. Case I, $x \vee y = 1$
$x \triangle_0 y = x \wedge y \leq x$
Similarly $x \triangle_0 y = x \wedge y \leq y$
Together, $x \triangle_0 y = (x \triangle_0 y) \triangle (x \triangle_0 y) \leq x \triangle y \leq x \triangle 1 = x$
Similarly, $x \triangle_0 y \leq x \triangle y \leq y$
Together, $\triangle_0 \leq \triangle \leq \triangle_5$
Case II $x \vee y = 0$
In this case, $x = y = 0$ so that the inequality trivially holds. ■

Definition 57 *A t-norm* $\triangle$ *is* ***convex*** *if whenever* $x \triangle y \leq c \leq x_1 \triangle y_1$*, then there is an* r *between* x *and* x_1 *and* s *between* y *and* y_1 *such that* $c = r \triangle s$

We will move to a more detailed generalisation of m when we consider fuzzy sets. For now, this definition is all that will be offered.

1.4 Order Theory

Definition 58 *Let* A *be a non-empty subset and* $R \subseteq A \times A$ *be a relation.* R *is* ***reflexive*** *if* $(a, a) \in R$ *for all* $a \in A$*. A relation* R *is called* ***symmetric*** *if* $(x, y) \in R$ *implies* $(y, x) \in R$*.* R *is called* ***transitive*** *if* $(x, y) \in R$ *and* $(y, z) \in R$ *implies* $(x, z) \in R$*. A relation obeying all three is called an* ***equivalence*** *relation.*

Exercise 59 *Let* R *be a binary relation on* A *and* $\triangle = \{(a, a) \mid a \in A\}$*. Show that* $\triangle$ *is reflexive and that* R *is reflexive if and only if* $\triangle \subseteq R$

Solution 60 *Since* $\triangle$ *is a collection of* (a, a) *for* $a \in A$*, the relation* $\triangle$ *is trivially reflexive.*

$(\implies)$ $(a, a) \in \triangle \implies (a, a) \in R$ *since* R *is reflexive.*

$(\impliedby)$ *If* $(a, a) \in \triangle \subseteq R$ *implies* $(a, a) \in R$*, which implies* R *is reflexive.*

Exercise 61 *Let* R *be a binary relation on* A *and define* $Inv\,(R) = \{(y, x) \mid (x, y) \in R\}$*. Show that* R *is symmetric if and only if* $Inv\,(R) \subseteq R$

Solution 62 $(\implies)$ $(y, x) \in Inv\,(R)$ *implies* $(x, y) \in R$ *by definition and* $(y, x) \in R$ *by symmetricity of* R*.*

$(\impliedby)$ *Let* $(x, y) \in R$*. Then,* $Inv\,(R) \subseteq R$ *implies that* $(y, x) \in Inv\,(R)$ *which implies and* $(y, x) \in R$

Exercise 63 *Show that* R *is transitive if and only if* $R \circ R \subseteq R$

Solution 64 $(\implies)$ *Let* $(x, z) \in R \circ R$*. Then, there exist* y *such that* $(x, y) \in R$ *and* $(y, z) \in R$*, which implies* $(x, z), (x, y) \in R$ *since* R *is transitive.*

$(\impliedby)$ *If* $(x, y) \in R$ *and* $(y, z) \in R$*, we have* $(x, z) \in R \circ R \implies (x, z) \in R$ *since* $R \circ R \subseteq R$ *by hypothesis.*

Definition 65 *A partition of a set* X *is a set* P *of cells or blocks that are subsets of* X *such that*

1. *If* $C \in P$ *then* $C \neq \varnothing$
2. *If* $C_1, C_2 \in P$ *and* $C_1 \neq C_2$ *then* $C_1 \cap C_2 = \varnothing$
3. *If* $a \in X$ *there exists* $C \in P$ *such that* $a \in C$

Definition 66 *If* R *is an equivalence relation on* X*, the* ***equivalence class*** *of* $a \in X$ *is the set* $[a] = \{b \in X \mid R(a, b)\}$

Lemma 67 $[a] = [b] \iff R(a, b)$

Proof. ($\Longrightarrow$)
Trivial
($\Longleftarrow$)
$R(a,b)$ and assume $[a] \neq [b]$. Then, $[a] \cap [b] = \varnothing \implies \not R(a,b)$ ■

Theorem 68 *The set of all equivalence classes under relation R form a partition of X, called X/R*

Proof. $[a] \in X/R$, then $R(a,a) \implies a \in [a] \implies [a] \neq \varnothing$

$[a],[b] \in X/R$ and $[a] \neq [b]$. Then, $\not R(a,b)$. Assume $x \in [a] \cap [b]$. Then, $R(x,a)$ and $R(x,b) \implies R(a,b)$. Contradiction. Thus $[a] \cap [b] = \varnothing$ ■

Definition 69 *A **partially ordered set**, (or poset) is a system $(P, \leq)$ where P is a non-empty set and $\leq$ is a binary relation on P satisfying, for all $x,y,z \in P$*

1. $x \leq x$
2. $x \leq y$ *and* $y \leq x$ *implies* $x = y$
3. $x \leq y$ *and* $y \leq z$, *then* $x \leq z$

Example 70 *Let X be a non-empty set. Then, $(\mathcal{P}(X), \subseteq)$ is a poset*

Example 71 *Let G be a group and $SubG$ the set of all subgroups of G. Then, $(SubG, \subseteq)$ is a poset*

Let H, K, L be subgroups. Then, since $H \subseteq H$, therefore $\subseteq$ is reflexive
If $H \subseteq K$ and $K \subseteq H$, then $H = K$ as sets and hence groups.
Finally, if $H \subseteq K$ and $K \subseteq L$, then $H \subseteq L$ as a subset and hence a subgroup.

If $Q \subseteq P$ and $\leq$ is restricted to members of $Q \times Q$, then $(Q, \leq_Q)$ is partially ordered.

Example 72 *Any non-empty collection Q of subsets of X ordered by containment forms a poset.*

Definition 73 *A partially ordered set is a **chain** or a **totally ordered set** if for every $x, y \in P$, $x \leq y$ or $y \leq x$*

Definition 74 *The system $(P, \leq)$ is an **anti-chain** if for any two distinct elements x and y, neither $(x,y) \in \leq$ nor $(y,x) \in \leq$*

In such a case, the only partial order definable is the equality relation.

Definition 75 *In a poset, x is covered by y, written $x \prec y$, if there does not exist $z \in P$ such that $x \leq z \leq y$*

In this case, unlike the usual understanding, $x \neq y$ and $y \neq z$. This covering relation determines the partial order for a finite set. In fact, the partial order is the smallest relation containing $\prec$.

Proof. Assume P is a finite poset. Suppose P is not determined by its covering relations. Then there exist $x, y \in P$ s.t. for all $w, z \in [x, y]$, w does not cover z. Here, $[x, y] := \{x, ..., y\}$ such that for any z in $[x, y]$ we have $x \leq z \leq y$. Choose $p_1 \in (x, y)$. Here, $(x, y) := \{x, ..., y\}$ such that for any z in (x, y) we have $x \leq z \leq y$ with $x \neq z$ and $y \neq z$. Such an element exists since y does not cover x. Since $[x, p_1] \subseteq [x, y]$, $[x, p_1]$ is not determined by its cover relations. Now choose $p_2 \in (x, p_1)$. Continuing inductively defines an infinite subset $\{p_1, p_2, p_3, ...\}$ of P, implying the contradiction P is infinite. Therefore, P is determined by its covering relations.

To prove that $\prec$ is the smallest covering relation. let $\prec_1$ and $\prec_2$ be two covering relations which determine the partial order. Let $x \prec_1 y$ and $y \prec_2 x$. Then, by the determined partial relation, $x = y$ so that $(x, y) \in \prec_1$ implies $(x, y) \in \prec_2$and conversely, so that $\prec_1 = \prec_2$ ■

Definition 76 *A mapping $f : (P, \leq_P) \longrightarrow (Q, \leq_Q)$ is called* ***order preserving*** *if $x \leq_P y$ implies $f(x) \leq_Q f(y)$*

Definition 77 *Two posest P and Q are* ***isomorphic****, written $P \cong Q$ if a bijective f and f^{-1} are order preserving maps between them.*

Theorem 78 *Let Q be a poset and let $\phi : Q \longrightarrow \mathcal{P}(Q)$ be defined by $\phi(x) = \{y \mid y \in Q \text{ and } y \leq x\}$. Then, $Q \cong \mathcal{R}(Q)$ ordered by $\subseteq$.*

Proof. By definition, $\phi : Q \longrightarrow \mathcal{R}(Q)$ is onto. Let $\phi(x_1) = \phi(x_2)$. Then $\{y \mid y \in Q \text{ and } y \leq x_1\} = \{y \mid y \in Q \text{ and } y \leq x_2\}$. That is, for any $a_1 \in \phi(x_1)$ and $a_2 \in \phi(x_2)$, $a_2 \in \phi(x_1)$ and $a_1 \in \phi(x_2)$. In particular, $x_1 \in \phi(x_2)$ and $x_2 \in \phi(x_1)$ since $x_2 \leq x_2$ and $x_1 \leq x_1$. Thus, we have $x_2 \leq x_1$ and $x_1 \leq x_2$. Since $\leq$ is a partial order, by anti-symmetricity, we have $x_2 = x_1$. Hence, ϕ is bijective.

Next, let $a \leq b$. Then,

$$\phi(a) = \{y \mid y \in Q \text{ and } y \leq a\}$$

and

$$\phi(b) = \{y \mid y \in Q \text{ and } y \leq b\}$$

By assumption $(a \leq b)$, we have $a \in \phi(b)$. For any $x \in \phi(a)$, we have $x \leq a$. By assumption, we also have $a \leq b$. Hence, by transitivity, we have $x \leq b$, implying $x \in \phi(b)$. In summary, for any $x \in \phi(a)$, we have $x \in \phi(b)$. Thus, $\phi(a) \subseteq \phi(b)$, implying ϕ is order preserving.

Define $\phi^{-1}(x) = b$ for $x = \phi(b)$. This is well-defined since ϕ is bijective. If $\phi(a) \subseteq \phi(b)$, then, by definition, $a \leq b$. Hence, for $x \subseteq y$, $\phi^{-1}(x) \leq \phi^{-1}(y)$, which implies ϕ^{-1} is order preserving, as well. ■

Definition 79 *Let P be a poset. Then, then $I \subseteq P$ is called an* ***ordered ideal*** *if for $x \in I$ and $y \leq x$, we have $y \in I$*

Definition 80 *Let P be a poset. Then, then $F \subseteq P$ is called an **ordered filter** if for $x \in F$ and $x \leq y$, we have $y \in I$*

The dual of I is F

Definition 81 *A poset P has a **maximum** or **greatest element** x if $x \geq y$ for all $y \in P$.*

Definition 82 *A poset P has a **minimum** or **least element** x if $y \geq x$ for all $y \in P$.*

The maximum is the dual of the minimum

Definition 83 *An element m of a poset P is called **minimal** if there is no $y \in P$ such that $y \leq m$ and $m \neq y$*

Definition 84 *An element m of a poset P is called **maximal** if there is no $y \in P$ such that $m \leq y$ and $m \neq y$*

The maximal is the dual of the minimal

Lemma 85 *The following are equivalent for a poset P:*

1. *Every non-empty subset $S \subseteq P$ contains an element minimal in S*
2. *P satisfies the decreasing chain condition, that is, P contains no infinite decreasing chain $a_0 > a_1 > a_2 > ...$*
3. *If $a_0 \geq a_1 \geq a_2 \geq ...$ in P, then there exists $k \in \mathbb{N}$ such that $a_n = a_k$ for all $n \geq k$.*

Proof. $(1 \implies 2)$

Let a_n be a minimal element. Then, if $a_0 > a_1 > a_2 > ... > a_n$, there does not exist a_{n+k} for $k \in \mathbb{N}$

$(2 \implies 3)$

If $a_0 > a_1 > a_2 > ... > a_n$, then $a_0 \geq a_1 > a_2 > ... > a_n$. Applying the principle of weakening n-times, we get $a_0 \geq a_1 \geq a_2 \geq ... \geq a_n$. Hence, if we have a_{n+1} different from a_j for $1 \leq j \leq n$, $a_n \ngeq a_{n+1}$, otherwise a decreasing chain would exist since we could proceed indefinitely. Hence, $a_{n+l} = a_n$ for $l \in \mathbb{N}$. Rephrased, this is $a_n = a_k$ for all $n \geq k$ where $k \in \mathbb{N}$

$(3 \implies 1)$

Suppose that there is no minimal element. Then, for any a_i, we can find $a_{i+1} < a_i$, which contradicts the finiteness of $a_0 \geq a_1 \geq a_2 \geq ...$ ■

Definition 86 *A poset P is said to satisfy the **ascending chain condition** (ACC) if every strictly ascending sequence of elements eventually terminates. Equivalently, given any sequence $a_0 \leq a_1 \leq a_2 \leq ...$, then there exists a positive integer k such that $a_n = a_k$ for all $n \geq k$ where $k \in \mathbb{N}$*

Definition 87 *A* ***propositional function*** $\phi(x_1, x_2, ...)$ *is an operator which acts on the objects denoted by the object variables* $x_1, x_2, ...$ *in a particular universe to return a truth value of false or true which depends on:*

1. *The values of* $x_1, x_2, ...$
2. *The nature of* ϕ

Theorem 88 (Subset of Set with Propositional Function) *Let* S *be a set. Let* $\phi : S \longrightarrow \{true, false\}$ *be a propositional function on* S*. Then,* $\{x \in S \mid \phi(x)\} \subseteq S$

Proof. $s \in \{x \in S \mid \phi(x)\} \implies s \in \{x \in S \wedge \phi(x)\} \implies s \in \{x \in S\} \implies s \in S \implies \{x \in S \mid \phi(x)\} \subseteq S$ ■

Theorem 89 (Strong Principle of Induction) *Let* $(P, \leq)$ *be a poset not satisfyng* ACC *and let* $\phi(x)$ *be a true statement for some* $x \in P$*. If 1)* $\phi(x)$ *holds for all minimal elements of* P *and 2)* $\phi(x) \implies \phi(y)$ *for all* $x \leq y$ *and* $y \neq x$*, then* $\phi(m)$ *holds for all* $m \in P$

Proof. Let $S = \{a \in P \mid \phi(a)\}$. That is, the set of all $a \in P$ for which $\phi(a)$ holds. Then, $S \subseteq P$. That is, the collection of all elements of S which satisfy ϕ is a subset of P. We have that $x \in S$ from hypothesis. Let $y \in P$. Now suppose that $x \leq x_1 \leq x_2 \leq ... \leq y \in S$. That is, $\phi(x), \phi(x_1), \phi(x_2), \ldots, \phi(y)$ all hold. Then that means $(P_y \backslash P_x) \subseteq S$ where $P_a := \{x \mid x \leq a\}$. From (2) it follows that $\phi(y_1)$ holds for $y_1 \geq y$, and so $(P_{y_1} \backslash P_x) \subseteq S$. Thus we have established: that

$S \subseteq P$

$x \in S$ and

$(P_y \backslash P_x) \subseteq S \implies (P_{y_1} \backslash P_x) \subseteq S$

We can continue this step for $y_n \geq y_{n+1}$ for $y_i \neq y_j$ where $n, i, j \in \mathbb{N}$. It follows that $(P \backslash P_x) \subseteq S$. That is, for every element $b \in P \backslash P_x$, it follows that $\phi(b)$ holds. But $P \backslash P_x$ is precisely the set of all $a \in P$ such that $b \geq x$. Hence the result. ■

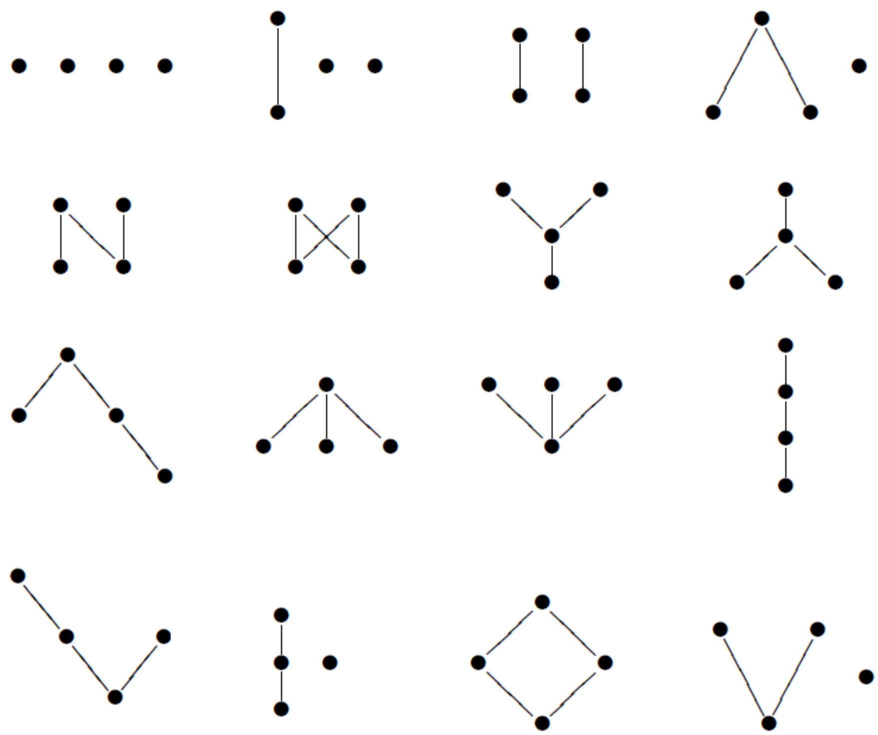
Exercise 90 *Draw the Hasse diagrams for all 4-element ordered posets.*

Exercise 91 *Let* $T : S \longrightarrow X$ *for* $S = \mathcal{D}(T)$ *is a subset of* X*. Define* $T \leq \Gamma$ *if* $\mathcal{D}(T) \subseteq \mathcal{D}(\Gamma)$ *and* $T(x) = \Gamma(x)$ *for all* $x \in \mathcal{D}(T)$*. Show that the collection of all partial maps on* X *is an ordered set.*

Solution 92 *Since trivially* $\mathcal{D}(T) \subseteq \mathcal{D}(T)$ *and* $T(x) = T(x)$*, we thus have* $T \leq T$

Next, if $T \leq \Gamma$ *and* $\Gamma \leq T$*, then* $\mathcal{D}(T) \subseteq \mathcal{D}(\Gamma)$ *and* $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(T)$*, which implies* $\mathcal{D}(\Gamma) = \mathcal{D}(T)$ *and* $T(x) = \Gamma(x)$ *for* $x \in \mathcal{D}(T) = \mathcal{D}(\Gamma)$*. Thus* $T \leq \Gamma$ *and* $\Gamma \leq T$ *implies* $\Gamma = T$*.*

4



$_1.png$

Finally, let $T \leq \Gamma$ and $\Gamma \leq \Psi$. Then, $\mathcal{D}(T) \subseteq \mathcal{D}(\Gamma)$ and $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Psi)$ and $T(x) = \Gamma(x)$ for $x \in \mathcal{D}(T)$ and $\Gamma(x) = \Psi(x)$ for $x \in \mathcal{D}(\Gamma)$. Now, $\mathcal{D}(T) \subseteq \mathcal{D}(\Gamma)$ and $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Psi)$ imply $\mathcal{D}(T) \subseteq \mathcal{D}(\Psi)$ and $T(x) = \Gamma(x)$ for $x \in \mathcal{D}(T)$ and $\Gamma(x) = \Psi(x)$ for $x \in \mathcal{D}(\Gamma)$ imply $T(x) = \Psi(x)$ for $x \in \mathcal{D}(T)$ which, by definition, is $T \leq \Psi$

Exercise 93 *Give an example of a map $f : (P, \leq_P) \longrightarrow (Q, \leq_Q)$ which is order preserving but not an isomorphism.*

Solution 94 *Let $(P, \leq_P) = \{(x,x), (y,y)\}$ and $(Q, \leq_Q) = \{(a,a), (b,b), (a,b)\}$. Define $f(x) = a$ and $f(y) = b$. Then, f is order preserving but not an isomorphism since the inverse of $a \leq_Q b$ is not present in the domain.*

Theorem 95 *$(P, \leq)$ and $(Q, \leq)$ be two posets. Then, the following are equivalent:*

1. *$P \cong Q$*
2. *There exists $f : (P, \leq) \twoheadrightarrow (Q, \leq)$ such that $f(x) \leq f(y)$ iff $x \leq y$*
3. *There exists $f : (P, \leq) \longrightarrow (Q, \leq)$ and $g : (Q, \leq) \longrightarrow (P, \leq)$, both order preserving such that $gf = I_P$ and $fg = I_Q$*

Proof. ($1 \implies 3$) Since $P \cong Q$, we can define $f : (P, \leq_P) \longrightarrow (Q, \leq_Q)$ and $f^{-1} = g : (Q, \leq_Q) \longrightarrow (P, \leq_P)$ where f and f^{-1} are bijective and order-preserving and $f^{-1}(q)$ if $f(p) = q$ for any $q \in Q$ and $p \in P$. Now, $fg(q) = f(p) = q$. Since this is valid for every q, $fg = I_Q$. Similarly, $g(f(p)) = g(q) = p$, from which we have $gf = I_P$.

($3 \implies 1$) Since the left and right inverse of f is g, f is bijective. Thus, $f^{-1} = g$ and both f and f^{-1} are order-preserving, implying $P \cong Q$

($1 \implies 2$) $P \cong Q$ implies there exist bijective $f : (P, \leq_P) \longrightarrow (Q, \leq_Q)$ such that $f(x) \leq f(y)$ whenever $x \leq y$. In particular, f is onto. Let $f(x) \leq f(y)$. Then, since f^{-1} is order-preserving, $f^{-1}f(x) \leq f^{-1}f(y)$ or $x \leq y$

($2 \implies 1$) $x, y \in f^{-1}(q_1)$ implies $f(x) = q = f(y)$ from which we have $f(x) \leq f(y)$ and $f(y) \leq f(x)$ if and only if $x \leq y$ and $y \leq x$, from which we have $x = y$. Thus, f is bijective. To show that f^{-1} is order preserving, take $f(x) \leq f(y)$, then, $x \leq y$ or $f^{-1}(f(x)) \leq f^{-1}(f(y))$ since $x = f^{-1}(f(x))$ ■

Theorem 96 *The following set-theoretic axioms are equivalent*

1. *(Axiom of Choice) If X is non-empty set, then there is a map $\phi : \mathcal{P}(X) \longrightarrow X$ such that $\phi(A) \in A$ for every non-empty set $A \subseteq X$*
2. *(Zermelo Well-ordering principle). Every non-empty set admits a well-ordering (a total order satisfying DCC)*
3. *(Hausdroff Maximality Principle) Every chain in a poset P can be embedded in a maximal chain*

4. *(Zorn's lemma) If every chain in a poset P has an upper bound in P, then P contains a maximal element*

5. *If every chain in a poset P has a least upper bound in P, then P contains a maximal element.*

Lemma 97 *Given a poset P and $a \nleq b$, there exists an extension $\leq^*$ of $\leq$ such that $(P, \leq)$ is a chain and $b \leq^* a$*

Proof. Let $a \nleq b$. Define

$$x \leq' y = \begin{cases} x \leq y \text{ or} \\ x \leq b \text{ or } a \leq y \end{cases}$$

Then, $x \leq' x$ holds. Also, if $x \leq' y$ and $y \leq' x$, then $x = y$. Transitivity also holds. Thus, $\leq'$ is a partial order with $b \leq' a$. Repeated application for this in the finite case yields a total order $\leq^*$. For the infinite case, apply Zorn's lemma (the union of a chain of partial orders is again a partial order) to obtain a total order $\leq^*$, extending $\leq$ ∎

Definition 98 *Let P be a poset and let $S \subseteq P$. $x \in P$ is an upper bound of S if $x \geq s$ for all $s \in S$. x is called the **least upper bound** or **supremum** of x is an upper bound and $x \leq x_n$ for all upper bounds x_n*

Theorem 99 *Every partial ordering on a set X is the intersection of total orders on X.*

Proof. Let R be a partial order on X, and let S be the set of all total orders which extend R. Since every total order is a partial order, the intersection of the orders in S certainly contains R. We show it is no bigger. So suppose that a and b are incomparable in R. Since there is a total order extending R in which $a \leq_1 b$, and another in which $b \leq_2 a$. So in the intersection of these total orders, a and b are still incomparable. ∎

1.5 Lattice Theory

Definition 100 *A **semilattice** is an algebra $S = (S, *)$ satisfying for all $x, y, z \in S$*

1. $x * x = x$

2. $x * y = y * x$

3. $x * (y * z) = (x * y) * z$

In other words, a semilattice is an idempotent commutative semigroup.

Example 101 *For a non-empty set X, $(\mathcal{P}(X), \cap)$ is a semi-lattice as is $(\mathcal{P}(X), \cup)$*

Theorem 102 *In a semi-lattice S, define $x \leq y$ if and only if $x * y = x$. Then, $(S, \leq)$ forms a poset in which every pair of elements has a greatest lower bound, denoted by $x * y$. Conversely, given an ordered set $(P, \leq)$ with the property that every pair of elements has a greatest lower bound. Define $x * y = \sup\{x, y\}$. Then, $(P, *)$ is a semi-lattice*

Proof. Since every semi-lattice is idempotent, we have $x \leq x$. Let $x \leq y$ and $y \leq x$. Then, $x * y = x$ and $y * x = y$. Combined,

$x = x * y = x * (y * x) = (x * y) * x = (y * x) * x = y * (x * x) = y * x = y$

Hence, $x = y$

Finally, let $x \leq y$ and $y \leq z$. Then, $x * y = x$ and $y * z = y$ and $x * z = (x * y) * z = x * (y * z) = x * y = x$. That is, $x * z = x$ so that $x \leq z$

If $x \leq y$, then the greatest lower bound of x and y, $x * y = x$. On the other hand, if $y \leq x$, then $y * x = x * y = y$

Finally, if $a \leq x$ and $a \leq y$, then, $a * x = a$ and $a * y = a$ from which the greatest lower bound of x and y, $a = a * x = x * a = x * (a * y) = x * (y * a) = (x * y) * a$

Assume that the glb of x, y, i.e. $x * y = a$ does not exist but then S is not a semi-lattice since the binary operation is not defined for x, y

Conversely, we show that $(P, \leq)$ is a semi-lattice.

1. $x * x = \sup\{x, x\} = \sup\{x\} = x$
2. $x * y = \sup\{x, y\} = \sup\{y, x\} = y * x$
3. $(x * y) * z = \sup\{\sup\{x, y\}, z\} = \sup\{x, y, z\} = \sup\{x, \sup\{y, z\}\} = x * (y * z)$ ■

Such a lattice in which the glb is defined is called a meet semi-lattice with $\wedge$ as the binary operation.

Definition 103 *A **homomorphism** between two semi-lattices $(S, *)$ and $(T, *')$ is a function $f : S \longrightarrow T$ such that $f(x * y) = f(x) *' f(y)$ for all $x, y \in S$. Two lattices are **isomorphic** if the homomorphism is bijective.*

Theorem 104 *Two semi-lattices are isomorphic if and only if they are isomorphic as ordered sets.*

Proof. $(\Longrightarrow)$

Let $(S, *)$ and $(T, *')$ be two semi-lattices and let f be an isomorphism between the semi-lattices. From a semi-lattice, we can define an ordered set by defining $x * y = x$ if and only if $x \leq y$ for all $x, y \in S$ and $x *' y = x$ if and only if $x \leq' y$ for all $x, y \in T$. Then, $x \leq y \implies x * y = x \implies f(x * y) = f(x) \implies f(x) *' f(y) = f(x) \implies f(x) \leq' f(y)$.

To prove that f^{-1} is also order preserving, let $f(x) \leq' f(y)$, then $f(x) *' f(y) = f(x) \implies f(x * y) = f(x)$. Since f is bijective, we have $x * y = y \implies x \leq y \implies f^{-1}(f(x)) \leq f^{-1}(f(y))$.

$(\Longleftarrow)$ Let $(S, \leq) \cong (T, \leq')$ under f. Define $x \wedge y = glb\{x, y\}$ to get a meet-lattice for $x, y \in S$. Then, $x \leq y \implies x \wedge y = x$ so that $f(x) \leq' f(y)$ implies $f(x \wedge y) = f(x) = f(x) \wedge' f(y)$. ■

Theorem 105 *The collection of all ordered ideals of a meet semi-lattice S forms a semi-lattice $O(S)$ under intersection*

Proof. Let $O(S)$ be the collection of all ordered ideals and let $I_1, I_2 \in O(S)$. Then, if $y_1 \in I_1$, $y_2 \in I_2$ and $x_1 \leq y_1$ and $x_2 \leq y_2$ implies $x_1 \in I_1$ and $x_2 \in I_2$. Let $x \in I_1 \cap I_2$. Then, $x \in I_1$ and $x \in I_2$. For $y \leq x$, $y \in I_1$ and $y \in I_2 \implies y \in I_1 \cap I_2$. Thus, the intersection of any two ideals $I_1 \cap I_2$ forms an ideal so that "$\cap$" is a binary operation. Now, for $I \cap I = I$, the idempotent law is trivially satisfied so any set.

Next, $I_1 \cap I_2 = I_2 \cap I_1$

The intersection of three sets is also associative. ■

Theorem 106 *Let S be a meet semi-lattice. Define $\phi : S \longrightarrow O(S)$ by $\phi(x) = \{y \in S \mid y \leq x\}$. Then, S is isomorphic to $(\phi(S), \cap)$*

Proof. As already proved, the set of ordered ideals of a semi-lattice S forms a semi-lattice $O(S)$. It remains to prove that ϕ is structure preserving and bijective. Let $y \leq x$. Then, $a \in \phi(y)$, $a \leq y \leq x \implies a \in \phi(x)$. Hence, $y \leq x \implies \phi(y) \subseteq \phi(x)$. In this case, $x \wedge y = x$ implies $\phi(x \wedge y) = \phi(x) = \phi(x) \cap \phi(y)$ since $\phi(y) \subseteq \phi(x)$. Similarly, $x \wedge y = y$ can be treated. Now, if $x \wedge y = z$ for some $z \in S$, then $z \leq x$ and $z \leq y$ so that $\phi(z) \subseteq \phi(y)$ and $\phi(z) \subseteq \phi(x) \implies \phi(z) \subseteq \phi(x) \cap \phi(y)$. Hence, $\phi(x \wedge y) = \phi(z) \subseteq \phi(x) \cap \phi(y)$. Note that $\phi(z)$ is the set of lower bounds of x and y. Now, let $c \in \phi(x) \cap \phi(y)$. Then, $c \leq z$ since c is a lower bound and z is the greatest lower bound. Hence, $\phi(x) \cap \phi(y) \subseteq \phi(z)$. Thus, $\phi(x) \cap \phi(y) = \phi(z)$ which implies $\phi(x \wedge y) = \phi(x) \cap \phi(y)$.

To prove that ϕ is bijective, first we prove that ϕ is one-to-one. Let $\phi(x) = \phi(y)$. Since $x \in \phi(x)$ (because $x \leq x$) and $y \in \phi(y)$. Therefore, $x \in \phi(y)$ and $y \in \phi(x)$. Thus, $x \leq y$ and $y \leq x$ which implies $x = y$.

Next, let $\{y \in S \mid y \leq x\}$ be an element of the image of ϕ. Then, $\phi^{-1}(\phi(x)) = \sup \phi(x)$. Since this is the dual of the meet operator, therefore $\sup \phi(x)$ must exist and hence for any element of the image of ϕ, we can find an element of the domain. ■

Definition 107 *A **lattice** is an algebra $\mathcal{L} = (L, \vee, \wedge)$ satisfying, for all $x, y, z \in L$*

1. $x \vee x = x$
2. $x \wedge x = x$
3. $x \vee y = y \vee x$
4. $y \wedge x = x \wedge y$
5. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
6. $(x \vee y) \vee z = x \vee (y \vee z)$

7. $x \wedge (x \vee y) = x$

8. $x \vee (x \wedge y) = x$

That is, a lattice is a meet-lattice and join-lattice with a way to connect both via the absorption law.

We have already seen isomorphism and homomorphism between ordered sets. Consider the lattices $(U, \vee_u, \wedge_u)$ and $(V, \vee_v, \wedge_v)$. Instead of a poset with one relation, we have two relations. In this case, we have the following definition:

Definition 108 *$f : U \longrightarrow V$ is a* ***homomorphism*** *of these two lattices if $f(x \wedge_u y) = f(x) \wedge_v f(y)$ and $f(x \vee_u y) = f(x) \vee_v f(y)$. f is an* ***isomorphism*** *if f is bijective. An isomorphism of a lattice with itself is an* ***automorphism****.*

Lemma 109 *Let $f : U \longrightarrow V$ be a homomorphism and $\sim \subseteq U \times U$ such that $a \sim b$ if $f(a) = f(b)$. Then, $\sim$ is an equivalence relation.*

Proof. $f(a) = f(a)$ so $\sim$ is reflexive

$f(a) = f(b)$ implies $f(b) = f(a)$ so that $\sim$ is symmetric.

Finally, $f(a) = f(b)$ and $f(b) = f(c)$ implies $f(b) = f(c)$ so that $a \sim b$ and $b \sim c$ implies $a \sim c$ ■

Lemma 110 *If $a \sim b$ and $c \sim d$, then $a \vee c \sim b \vee d$ and $a \wedge c \sim b \wedge d$*

Proof. By hypothesis, $f(a) = f(b)$ and $f(c) = f(d)$

$\Longrightarrow f(a) \vee f(c) = f(b) \vee f(d)$ and $f(a) \wedge f(c) = f(b) \wedge f(d)$

$\Longrightarrow f(a \vee c) = f(b \vee d)$ and $f(a \wedge c) = f(b \wedge d)$

$\Longrightarrow a \vee c \sim b \vee d$ and $a \wedge c \sim b \wedge d$ ■

Thus, this equivalence relation has two additional properties. Such a relation is called a congruence relation, which gives rise to homomorphisms.

Theorem 111 *If $\sim$ is a congruence on the lattice U, then the set of equivalence classes $U/\sim$ forms a lattice under the operation $[a] \vee [b] = [a \vee b]$ and $[a] \wedge [b] = [a \wedge b]$. The mapping $g : U \longrightarrow U/\sim$ such that $g(a) = [a]$ is a lattice homomorphism.*

Proof. Idempotent

$[a] \vee [a] = [a \vee a] = [a]$

$[a] \wedge [a] = [a \wedge a] = [a]$

Commutative

$[a] \vee [b] = [a \vee b] = [b \vee a] = [b] \vee [a]$

$[a] \wedge [b] = [a \wedge b] = [b \wedge a] = [b] \wedge [a]$

Associcative

$[a] \vee ([b] \vee [c]) = [a] \vee ([b \vee c]) = [a] \vee [(b \vee c)]$

$= [a \vee (b \vee c)]$

$= [(a \vee b) \vee c] = [(a \vee b)] \vee [c]$

$= ([a \vee b]) \vee [c]$

$[a] \wedge ([b] \wedge [c]) = [a] \wedge ([b \wedge c]) = [a] \wedge [(b \wedge c)]$

$= [a \wedge (b \wedge c)]$
$= [(a \wedge b) \wedge c] = [(a \wedge b)] \wedge [c]$
$= ([a \wedge b]) \wedge [c]$
Absorption laws
$[a] \wedge ([a] \vee [b])$
$= [a] \wedge [a \vee b]$
$= [a \wedge (a \vee b)]$
$= [a]$
$[a] \vee ([a] \wedge [b])$
$= [a] \vee [a \wedge b]$
$= [a \vee (a \wedge b)]$
$= [a]$
$g(a \vee b) = [a \vee b] = [a] \vee [b] = g(a) \vee g(b)$
$g(a \wedge b) = [a \wedge b] = [a] \wedge [b] = g(a) \wedge g(b)$ ■

Definition 112 *An isomorphism of a system with itself is called an* ***automorphism****.*

Theorem 113 *Let $\mathbb{I} = (\{0,1\}, \leq)$ be a poset and let Aut($\mathbb{I}$) be the set of all automorphisms of $\mathbb{I}$. Show that Aut($\mathbb{I}$) is a group with respect to composition of functions. This group is called the group of automorphisms of $\mathbb{I}$.*

Proof. Let $f, g \in$Aut($\mathbb{I}$). Then, $f(g(a \vee b)) = f(g(a) \vee g(b)) = f(g(a)) \vee f(g(b))$. Similarly, $f(g(a \wedge b)) = f(g(a)) \wedge f(g(b))$ so that Aut($\mathbb{I}$) is closed under composition. Function composition is trivially associative. Also, the identity map I is an automorphism since $I(a \vee b) = a \vee b = I(a) \vee I(b)$ and $I(a \wedge b) = I(a) \wedge I(b)$ so that the identity exists. Since any $f \in$Aut($\mathbb{I}$) is bijective, f^{-1} must exist. Now, $f(a \vee b) = f(a) \vee f(b)$ implies $f^{-1}f(a \vee b) = a \vee b = f^{-1}(f(a) \vee f(b))$. Let $f(a) = x$ and $f(b) = y$. Then, we have $\left(f^{-1}(x)\right) \vee \left(f^{-1}(y)\right) = f^{-1}(x \vee y)$. Similarly for the second binary operation so that $f^{-1} \in$Aut($\mathbb{I}$) ■

Definition 114 *Let $\diamond$ and $\circ$ be two t-norms. The systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \diamond)$ are* ***isomorphic*** *if there is an element $h \in$Aut($\mathbb{I}$) such that $h(x \diamond y) = h(x) \circ h(y)$. In such a case, the t-norms are said to be* ***isomorphic***

Isomorphism between t-norms is an equivalence relation and paritions *t*-norms into equivalence classes.
Proof. Let $(\mathbb{I}, \circ) \sim (\mathbb{I}, \diamond) \iff h(x \diamond y) = h(x) \circ h(y)$
Then, $h(x \diamond y) = h(x) \diamond h(y) \iff (\mathbb{I}, \diamond) \sim (\mathbb{I}, \diamond)$
Next, $(\mathbb{I}, \diamond) \sim (\mathbb{I}, \circ)$
$\iff h(x \diamond y) = h(x) \circ h(y)$
$\iff h(x \diamond y) = h(x \circ y)$
$\iff h(x) \diamond h(y) = h(x \circ y)$
$\iff (\mathbb{I}, \diamond) \sim (\mathbb{I}, \circ)$
Finally, If $(\mathbb{I}, \diamond_1) \sim (\mathbb{I}, \diamond_2)$ and $(\mathbb{I}, \diamond_2) \sim (\mathbb{I}, \diamond_3)$
then $h(x \diamond_1 y) = h(x) \diamond_2 h(y)$ and $h(x \diamond_2 y) = h(x) \diamond_3 h(y)$

Or $h(x \diamond_1 y) = h(x \diamond_2 y)$ and and $h(x \diamond_2 y) = h(x) \diamond_3 h(y)$
Therefore, $h(x \diamond_1 y) = h(x) \diamond_3 h(y)$
$\iff (\mathbb{I}, \diamond_1) \sim (\mathbb{I}, \diamond_3)$
Any equivalence relation partitions a set into equivalence classes. ∎

The t-norm min is rather special since it is the only idempotent t-norm and not isomorphic to any other so it is an equivalence class all by itself.

Corollary 115 *The set of automorphisms of* $(\mathbb{I}, \circ)$ *is a subgroup of* $Aut(\mathbb{I})$.

Proof. Let $h, g \in$Aut$(\mathbb{I}, \circ)$ such that $h(x \circ y) = g(x) \circ g(y)$. Then, $g^{-1}h(x \circ y) = x \circ y$ so that
$gh^{-1} \in$Aut$(\mathbb{I}, \circ)$ ∎

Thus, with each t-norm $\circ$, there is a group associated with it, namely the automorphism group Aut$(\mathbb{I}, \circ) = \{f \in Aut(\mathbb{I}) \mid f(x \circ y) = f(x) \circ f(y)\}$. This is the automorphism of the group of the t-norm $\circ$. For the t-norm min, it is clear that Aut$(\mathbb{I}, \min)$ =Aut$(\mathbb{I})$

If H is a subgroup of a group G and $g \in G$, then $g^{-1}Hg = \{g^{-1}hg \mid h \in H\}$ is a subgroup of G. This subgroup is said to be conjugate to H or a conjugate of H. The map $h \longrightarrow g^{-1}hg$ is an isomorphism from H to its conjugate $g^{-1}Hg$

Theorem 116 *If two t-norms are isomorphic, then their automorphism groups are conjugate*

Proof. Suppose that the systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \diamond)$ are isomorphic. Then, there is an element $f \in$Aut$(\mathbb{I})$ $f : (\mathbb{I}, \circ) \longrightarrow (\mathbb{I}, \diamond)$ such that $f(x \circ y) = f(x) \diamond f(y)$. The map $g \longrightarrow f^{-1}gf \in$Aut$(\mathbb{I})$ from Aut$(\mathbb{I}, \diamond)$ to Aut$(\mathbb{I}, \circ)$ so that f^{-1}Aut$(\mathbb{I}, \diamond) f$ =Aut$(\mathbb{I}, \circ)$ ∎

Theorem 117 *The meet and join operators in a lattice induce the same order*

Proof. Let $\leq$ and $\leq'$ be two orders induced by $\wedge$ and $\vee$, respectively. Define $x \vee y = y \iff x \leq y$ and $x \wedge y = x \iff x \leq' y$.

We have already proved that a semi-lattice forms a poset. Hence the definition makes sense.

Let $(x, y) \in\leq$. Then, from $x \vee y = y$ and $x \wedge (x \vee y) = x$
$x \wedge y = x \implies x \leq' y \implies (x, y) \in\leq'$

Conversely, let $(x, y) \in\leq'$. Then, from $x \wedge y = x$ and $x \vee (x \wedge y) = x$, we have
$x \vee y = (x \wedge y) \vee y = y \vee (x \wedge y) = y \vee (y \wedge x) = y$. That is, $x \vee y = y \implies (x, y) \in\leq$

Thus, $\leq=\leq'$ ∎

For a subset A of a partially ordered set $(P, \leq)$, let A^u denote the set of all upper bounds of A.That is, $A^u = \{x \in P \mid x \geq a, \forall a \in A\}$. Similarly we can define the set all lower bounds of A by $A^l \{x \in P \mid x \leq a, \forall a \in A\}$.

When does A have a least upper bound and greatest lower bound? A^l and A^u are non-empty if the poset is bounded. Thus, for any $l \in A^l$ and $u \in A^u$, $a \leq u$ and $l \leq a$ for every $a \in A$. The least upper bound of A exists when the greatest lower bound of A^u exists. Similarly, the greatest lower bound of A exists when the least upper bound of A^l exists.

In such a case, $\sup A = \bigvee A = \bigwedge A^u$ and $\inf A = \bigwedge A = \bigvee A^l$.

Theorem 118 *Let S be a finite meet-lattice with greatest element 1. Then, S is a lattice with the join defined by $x \vee y = \bigwedge \{x, y\}^u$*

Proof. By hypothesis, we have $x \wedge x = x$, $x \wedge y = y \wedge x$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$. Note that for any x, y, $\{x, y\}^u$ is non-empty since $1 \geq x, y$. Also, $\bigwedge \{x, y\}^u$ will always exist since we have a semi-lattice. It follows that if $\bigwedge \{x, y\}^u = a$, then $a \leq a_i$ for all $a_i \in \{x, y\}^u$ so that $a = x \vee y$. Hence, the definition makes sense.

Now, $x \vee x = \bigwedge \{x, x\}^u = \bigwedge \{x\}^u = a$ (say)

By definition, $a \geq x$ and for any $z \in \{x\}^u$, $a \leq z$

Thus, $x \geq x \implies x \in \{x\}^u$ and for any $z \in \{x\}^u$, $x \leq z$

$\implies x = \bigwedge \{x\}^u = x \vee x$

Next, $x \vee y = \bigwedge \{x, y\}^u = \bigwedge \{y, x\}^u = y \vee x$

For associativity, $x \vee (y \vee z) = \wedge \{x, \wedge \{y, z\}^u\}^u$

If $\bigwedge \{y, z\}^u = a$ (say) and $\bigwedge \{x, a\}^u = b$

These exist because $y \vee z$ and consequently $a \vee x$ exists

Then, $a \geq y, z$ and $b \geq x, a$ and for any $a_i \in \{y, z\}^u$ and $b_i \in \{x, a\}^u$, $b \leq b_i$ and $a \leq a_i$

Now, $a \geq y, z$ and $b \geq x, a \implies b \geq x, y, z$ imply $a \wedge b \leq \{x, y, z\}^u$, therefore $\bigwedge \{x, \wedge \{y, z\}^u\}^u = \bigwedge \{x, y, z\}^u$ and so $\wedge$ is associative

To prove that the absorption laws hold, $y \wedge (y \vee x) = y \wedge (\bigwedge \{x, y\}^u) =$ Let $\bigwedge \{x, y\}^u = a$. Then, $a \leq a_i$ where $a_i \in \{x, y\}^u$. In particular, $x \leq x$ and $y \leq y$ implies $x, y \in \{x, y\}^u$ so that $a = x$ or y. If $a = x$, then $y \wedge a = x$ but this is not possible since we have defined $x \leq y \iff x \vee y = y$. Thus, $a = y$ is the only possibility. Hence, $y \leq x$ so that $y \wedge y = y$

For the second part, $x \vee (x \wedge y) = \bigwedge \{x, x \wedge y\}^u$. If $x \wedge y = a$, then $\bigwedge \{x, a\}^u = x$ since the upper bounds of x, a include x by definition of $x \wedge y = a$ and $x \in \{x, a\}^u$, we must have $x \vee (x \wedge y) = \bigwedge \{x, x \wedge y\}^u = x$. ■

This result not only yields an immediate supply of lattice examples but it provides us with an efficient algorithm for deciding when a finite ordered set is a lattice. If P has a greatest element and every pair of elements has a meet, then P is a lattice.

The dual version says that if every join-lattice has a smallest element, then that join-lattice is a lattice.

Every finite subset of a lattice has a greatest and least upper bound but these bounds need not exist for infinite subsets. For instance, the set of rational numbers with the usual ordering is not bounded above and hence does not have a greatest element and thus no greatest upper bound.

Definition 119 *A lattice is said to be* ***complete*** *if for every subset A of the lattice, $\bigvee A$ and $\bigwedge A$ exists.*

Remark 120 *Every finite lattice is complete*

Proof. In a lattice, the meet and join operations are defined for every two elements. Since any subset of a finite set is finite, therefore we can define the meet of any finite subset A of lattice L as follows: if $A = \{x_1, x_2, ..., x_n\}$, then $\vee A = x_1 \vee x_2 \vee ... \vee x_n$ with the brackets ignored since $\vee$ is associative. Similarly, $\bigwedge A = x_1 \wedge x_2 \wedge ... \wedge x_n$. Thus, the meet and join of any subset of a lattice exists, making the lattice complete. ■

Proposition 121 *Every complete lattice has a greatest and least element.*

Proof. Since for any complete lattice L, $L \subseteq L \implies \bigvee L$ and $\bigwedge L$ exists. To prove that $\bigwedge L = 0$ and $\bigvee L = 1$, assume $\bigwedge L \neq 0$ and $\vee L \neq 1$. Let $\bigwedge L = y$ and $\bigvee L = x$. Then, $y \leq x_i \ \forall i$ but then y is the least element $\implies y = 0$. Similarly, $x_i \leq x \ \forall i \implies x = 1$. ■

The converse is not generally true. For instance, the open sets of a topological space, ordered by inclusion, is a lattice. The supremum is given by the union of open sets and the infimum by the interior of the intersection. This forms a complete and bounded lattice. On the other hand, if we define infimum to be set intersection, the open sets form a bounded but not complete lattice since, in general, arbitrary intersections of open sets are not open. A simpler example would be as follows: Let $P \subset \mathbb{Q}$ with the usual order among rationals, $-q \leq p \leq q$ for all $p \in P$ for some $q \in \mathbb{Q}$. This is a lattice, with operations $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. In fact, this lattice gives rise to a totally ordered. Every finite subset of P has a least upper bound (the maximum) and a greatest lower bound (the minimum). However, the set $\{x \mid x \in P, \ x^2 < 2\}$ has no least upper bound.

By convention, $\bigwedge \varnothing = 1$ and $\bigvee \varnothing = 0$.

Definition 122 *An element q of a lattice $\mathcal{L}$ is called **join irreducible** if $q = \vee F$ for a finite set F implies $q \in F$.*

In essence, this states that q cannot be formed by considering the join of some other elements. If that is the case, then q is among the elements. The set of join irreducible elements in L is denoted by $J(\mathcal{L})$.

Proposition 123 $0 \in J(\mathcal{L})$

Proof. $\bigvee \varnothing = 0$ implies $0 \in \varnothing$ ■

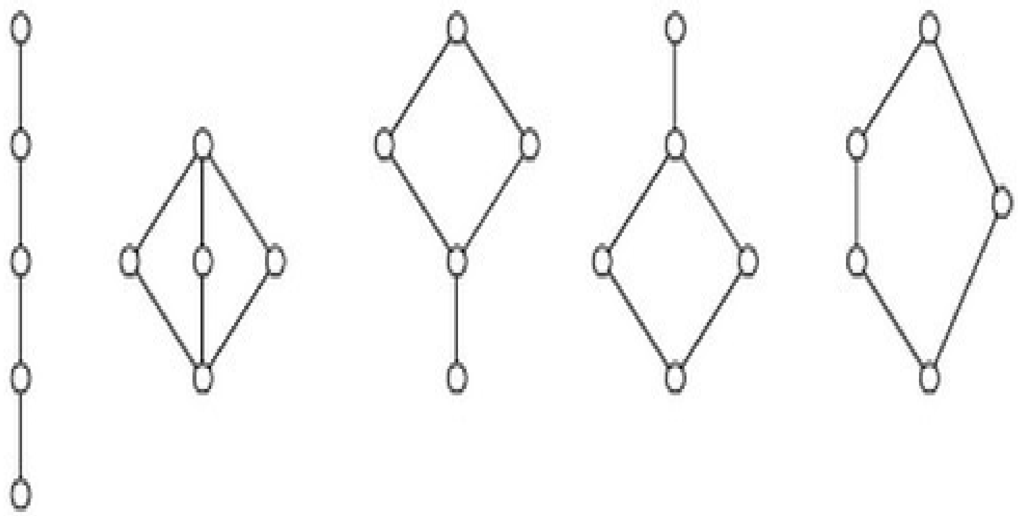
Lemma 124 *If a lattice $\mathcal{L}$ satisfies DCC, then every element of $\mathcal{L}$ is a join of finitely many joint irreducible elements.*

Proof. Since the join operation is defined for each element, we can always have $x = y \vee z$ where $y \leq x$ and $z \leq x$. If y and z are both join irreducible, then we are done. Otherwise, we can always write y or z as $a \vee b$ and repeat the argument. Since $\mathcal{L}$ has no infinite decreasing chains, this process ends after a finite number of steps. ■

Exercise 125 *Draw the Hasse diagrams for all 5 element meet-lattices*

Solution 126

5



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Exercise 127 *Draw the Hasse diagram for all 6 element lattices*

Exercise 128 *Draw the lattice of subspaces of the vector space* $\mathbb{R}^2$

Solution 129 *The only possible subspaces of* $\mathbb{R}^2$ *are* $\mathbb{R}^2$, $\varnothing$ *and lines* $y = kx$ *passing through the origin where* k *is any constant. Containment is easy to see and the intersection of any two subspaces is the trivial subspace and that the join, the span of two subspaces, is the* $\mathbb{R}^2$ *if the trivial subspace is ignored.*

Exercise 130 *Prove that a lattice which has a least element and satisfies the ACC is complete*

Proof. Let $0 \leq x_1 \leq x_2 \leq x_3 \leq ... \leq x_n \leq x_{n+1} \leq ...$

Then, $x_{n+i} = x_n \ \forall i$ and we have finite elements. The presence of the least element ensures that the set of lower bounds of any two elements is non-empty, making the meet of any two elements possible. The presence of a greatest element x_n ensures that the set of upper-bounds of any two elements is possible. Since we have a finite lattice, the subset of any finite lattice will have the meet and joined defined, making the lattice complete ■

Exercise 131 *Give explicitly the subgroup lattice for the cyclic group* $\mathbb{Z}_4$.

Solution 132 *The subgroups of* $\mathbb{Z}_4$ *are* $\{0\}, \{0, 2\}$ *and* $\mathbb{Z}_4$. *In this case, the simple union and intersection of sets can be used to define the meet and join.*

Exercise 133 *Let X,Y be sets and $R \subseteq X \times Y$ be a relation. For $A \subseteq X$ and $B \subseteq Y$, let*

$$\begin{aligned} \sigma(A) &= \{y \in Y \mid aRy \text{ for all } a \in A\} \\ \pi(B) &= \{x \in X \mid xRb \text{ for all } b \in B\} \end{aligned}$$

Prove that (a) $A \subseteq \pi\sigma(A)$ and $B \subseteq \sigma\pi(B)$ for all $A \subseteq X$ and $B \subseteq Y$, (b) $A \subseteq A' \implies \sigma(A) \supseteq \sigma\left(A'\right)$ and $B \subseteq B' \implies \pi(B) \supseteq \pi\left(B'\right)$ and (c) $\sigma(A) = \sigma\pi\sigma(A)$ and $\pi(B) = \pi\sigma\pi(B)$ for all $A \subseteq X$ and $B \subseteq Y$

Solution 134 *(a) Let $a \in A \subseteq X$. Then, $y \in \sigma(A) \subseteq Y$, aRy. Using $\pi(\sigma(A)) = \{x \in X \mid xRy \text{ for all } y \in \sigma(A)\}$*

and aRy, we have $a \in \pi(\sigma(A))$

Similarly, let $b \in B \subseteq Y$. Then, $x \in \pi(B) \subseteq X$ implies xRb. Since $\sigma\pi(B) = \{y \in Y \mid xRy \text{ for all } x \in \pi(B)\}$, therefore $b \in \sigma\pi(B)$

(b) Let $x \in \sigma(A')$. Then, $\forall a' \in A$, $a\prime Ra$

$\implies$ *$a\prime Ra$ $\forall a' \in A'$ because $A \subseteq A'$*

$\implies x \in \sigma(A)$

$\implies \sigma(A) \supseteq \sigma\left(A'\right)$

Similarly $B \subseteq B' \implies \pi(B) \supseteq \pi\left(B'\right)$

(c) Let $\pi\sigma(A) = A'$ and $\sigma\pi(B) = B'$. Then, by (a) and (b), $\sigma(A) \supseteq \sigma\pi\sigma(A)$ and $\pi(B) \supseteq \pi\sigma\pi(B)$

Clearly, $x \in \sigma(A) \implies x \in \sigma\pi\sigma(A)$ by (b)

$\implies \sigma(A) \subseteq \sigma\pi\sigma(A)$

Similarly, $\pi(B) \subseteq \pi\sigma\pi(B)$, completing the proof.

Definition 135 *A lattice $\mathcal{L} = (L, \vee, \wedge)$ is* ***modular*** *if $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all $x, y, z \in L$*

Theorem 136 *A lattice $\mathcal{L} = (L, \vee, \wedge)$ is modular iff $x \leq z$ implies $x \vee (y \wedge z) \geq (x \vee y) \wedge z$*

Proof. ($\implies$) From modularity, $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all $x, y, z \in L$ so that we have $x \vee (y \wedge z) \geq (x \vee y) \wedge z$ and $x \vee (y \wedge z) \leq (x \vee y) \wedge z$

($\Longleftarrow$) $x \leq z$ and $x \leq x \vee y$ implies x is a lower bound for $\{z, x \vee y\}$. Also, $z \geq y \wedge z$ trivially for any lattice. Similarly, $y \geq y \wedge z \implies x \vee y \geq y \wedge z$. Thus, the set of lower bounds for $\{z, x \vee y\}^l \supseteq \{x, y \wedge z\}$. In effect, $x \vee (y \wedge z) \leq (x \vee y) \wedge z$, so that this and the hypothesis $x \vee (y \wedge z) \geq (x \vee y) \wedge z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ for any pair of elements $x, y, z \in L$ ■

Example 137 *Let M be a left R-module and L be the collection of all submodules of M. Then, L is a modular lattice*

In this case, $\wedge$ is replaced by the usual intersection. The ordering is the usual set-theoretic inclusion. The intersection of two submodules is a submodule. The greatest element here is M itself since it is a submodule of itself. Likewise, the intersection of all the submodules in L will give us the least element. $\vee$ is replaced by the span of two submodules. More rigorously, for $A, B \in L = A \vee B = A + B = \{a + b \mid a \in A, b \in B\}$. In fact, this is the smallest submodule containing A and B because if C is a submodule containing A and B, it is closed under addition. Thus for all $a \in A$ and $b \in B$, $a, b \in C$, hence $a + b \in C$ and $A+B \subseteq C$. Therefore $A+B$ "is the smallest" (under inclusion). Let $A, B, C \in L$ such that $A \subseteq C$. We need to show that $A + (B \cap C) \supseteq (A + B) \cap C$. Let $x \in (A + B) \cap C$. Then, $x = a + b$ and $a + b \in C$ for some $a \in A \subseteq C$ and $b \in B$. Then, $b = x - a$. Since $x \in C$ and $a \in C$, then $b = x - a \in C$. Hence, $b \in B \cap C$ and $x = a + b \in A + (B \cap C)$ so that $A + (B \cap C) \supseteq (A + B) \cap C$. Even though this suffices, we will show that $A + (B \cap C) \subseteq (A + B) \cap C$ holds, which is trivial in any lattice if $A \subseteq C$. Let $x \in A + (B \cap C)$. Then, $x = a + b$ where $b \in B \cap C$. Then, $x - a \in B$ and $x - a \in C$ and $x \in A + B$. Since $a \in A \subseteq C$ and $x - a \in C$, then $x \in C$. Thus, we have $x \in C$ and $x \in A + B \implies x \in (A + B) \cap C$.

Theorem 138 *Every totally ordered set is a modular lattice.*

Proof. We can form a lattice from the relation $\leq$ by relying on the fact that we can have a meet-lattice from $a \leq c \iff a \wedge c = a$. Similarly, $a \leq c \iff a \vee c = c$ to get a join-lattice. Using these two, we can prove the absorption laws as follows: Since we can have $a \leq c$, then $a \wedge (a \vee c) = a \wedge c = a$. $a \vee (a \wedge c) = a \vee a = a$. We can also have $c \leq a$ and then $a \wedge (a \vee c) = a \wedge a = a$ and $a \vee (a \wedge c) = a \vee c = a$. We have proved that from a totally ordered set, we can have a lattice. To prove that the lattice is modular, let $a \leq c$. We will prove that $a \vee (b \wedge c) = (a \vee b) \wedge c$ by arguing on a case-by-case basis

Case-I

$a \leq c \leq b$

Then, $a \vee (b \wedge c) = a \vee c = c$ and $(a \vee b) \wedge c = b \wedge c = c$

Case-II

$a \leq b \leq c$

$a \vee (b \wedge c) = a \vee b = b$ and $(a \vee b) \wedge c = b \wedge c = b$

Case-III

$b \leq a \leq c$

$a \vee (b \wedge c) = a \vee b = a$ and $(a \vee b) \wedge c = a \wedge c = a$

These are the only three possibilities. Any other possibility will reduce to either one of the case because of transitivity. ■

Exercise 139 *Show that a lattice $\mathcal{L} = (L, \vee, \wedge)$ is modular iff the equality $x \vee (y \wedge (x \vee t)) = (x \vee y) \wedge (x \vee t)$ holds*

Solution 140 $(\implies)$ *$x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all $x, y, z \in L$*

Since we can write $z = x \vee t$, then we are done.

$(\impliedby)$ *assume $x \vee (y \wedge (x \vee t)) = (x \vee y) \wedge (x \vee t)$. Then, $x \leq x \vee t = z$ (say) trivially and $x \vee (y \wedge (x \vee t)) = (x \vee y) \wedge (x \vee t)$ by hypothesis so that $x \vee (y \wedge z) = (x \vee y) \wedge z$*

Exercise 141 *Show that a lattice $\mathcal{L} = (L, \vee, \wedge)$ is modular iff $x \leq t$ and $z \leq y$ implies $x \vee (y \wedge (z \vee t)) = ((x \vee y) \wedge z) \vee t$*

Solution 142 $(\implies)$ $x \le t \implies x \vee (y \wedge t) = (x \vee y) \wedge t$
$z \le y \implies z \vee (t \wedge y) = (z \vee t) \wedge y$
$\implies x \vee (z \vee (t \wedge y)) = x \vee (y \wedge (z \vee t))$
Focusing on the left side, $x \vee (z \vee (t \wedge y))$
$= x \vee ((z \vee t) \wedge y)$
$= x \vee (y \wedge (z \vee t))$
$(\impliedby)$

Exercise 143 *Show that a lattice* $\mathcal{L} = (L, \wedge, \vee)$ *is modular iff* $a \wedge b = a \wedge c, a \vee c = a \vee b$, $b \le c$ *together imply* $b = c$ *for any* $a, b, c \in L$

Proof. $(\implies)$ From the modulartiy of $\mathcal{L}$, $b \le c$ implies $b \vee (a \wedge c) = (b \vee a) \wedge c$
Now, $b = b \vee (a \wedge b) = b \vee (a \wedge c) = (b \vee a) \wedge c = (a \vee c) \wedge c = c$
$(\impliedby)$ $b = b \vee (a \wedge b) = b \vee (a \wedge c)$
$c = (a \vee c) \wedge c = (b \vee a) \wedge c$
In effect, $b \vee (a \wedge c) = (b \vee a) \wedge c$ from $a \wedge b = a \wedge c, a \vee c = a \vee b$, $b \le c$ ■

Definition 144 *A lattice is* ***distributive*** *if either i)* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ *or ii)* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Proposition 145 *The above two conditions are equivalent*

Proof. i) implies ii)
$(a \wedge b) \vee (a \wedge c)$
$= ((a \wedge b) \vee a) \wedge ((a \wedge b) \vee c)$
$= a \wedge ((a \wedge b) \vee c)$
$= a \wedge (c \vee (a \wedge b))$
$= a \wedge ((c \vee a) \wedge (c \vee b))$
$= (a \wedge (c \vee a)) \wedge (c \vee b)$
$= a \wedge (c \vee b)$
$= a \wedge (b \vee c)$
ii) implies i)
$(a \vee b) \wedge (a \vee c)$
$= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c)$
$= a \vee ((a \vee b) \wedge c)$
$= a \vee ((a \wedge c) \vee (c \wedge b))$
$= (a \vee (a \wedge c)) \vee (c \wedge b)$
$= a \vee (c \wedge b)$ ■

Lemma 146 *Every distributive lattice is modular*

Proof. Let $x \le z$. Then, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) = (x \vee y) \wedge z$ ■
The converse is not true. Example, M_5 is modular but not distributive

Exercise 147 *Show that a lattice is distributive iff* $a \wedge b = a \wedge c, a \vee c = a \vee b$ *together imply* $b = c$ *for any* $a, b, c \in L$

Proof. ($\Longrightarrow$) Distributivity implies modularity so that this part can be proved using the previous exercise.

($\Longleftarrow$) $(a \vee b) \wedge (a \vee c)$
$= (a \vee c) \wedge (a \vee c)$
$= (a \vee c)$
$= a \vee (c \wedge c)$
$= a \vee (c \wedge b)$ ■

Definition 148 *Let $\mathcal{L} = (L, \wedge, \vee)$ be a lattice with a greatest 1 and least 0 element. A* ***complement*** *of an element a of L is an element a' of L such that $a \wedge a^{'} = 0$ and $a \vee a^{'} = 1$*

Proposition 149 *In a distributive lattice with a least element and greatest element*

(a) an element has at most one complement

(b) if $a^{'}$ is a complement of a and $b^{'}$ is the complement of b, then $a^{'} \vee b^{'}$ is the complement of $a \wedge b$ and $a^{'} \wedge b^{'}$ is the complement of $a \vee b$

Proof. (a) Suppose an element a has two complements a_1 and a_2. Then, $a \vee a_1 = 1 = a \vee a_2$

Similarly, $a \wedge a_1 = a \wedge a_2$. By the previous exercise, $a_1 = a_2$

(b) $\left(a^{'} \vee b^{'}\right) \wedge (a \wedge b)$
$= \left[\left(a^{'} \vee b^{'}\right) \wedge a\right] \wedge b$
$= \left[\left(a^{'} \wedge a\right) \vee \left(b^{'} \wedge a\right)\right] \wedge b$
$= b^{'} \wedge a \wedge b$
$= b^{'} \wedge b \wedge a$
$= 0 \wedge a$
$= 0$
$\left(a^{'} \vee b^{'}\right) \vee (a \wedge b)$
$= a^{'} \vee \left(b^{'} \vee (a \wedge b)\right)$
$= a^{'} \vee \left(\left(b^{'} \vee a\right) \wedge \left(b \vee b^{'}\right)\right)$
$= a^{'} \vee \left(\left(b^{'} \vee a\right) \wedge 1\right)$
$= a^{'} \vee b^{'} \vee a$
$= a^{'} \vee a \vee b^{'}$
$= 1 \vee b^{'}$
$= 1$

Similarly, $\left(a^{'} \wedge b^{'}\right) \vee (a \vee b)$
$= \left(\left(a^{'} \wedge b^{'}\right) \vee a\right) \vee b$
$= \left(\left(a^{'} \vee a\right) \wedge (b \vee a)\right) \vee b$
$= (1 \wedge (b' \vee a)) \vee b$
$= (b' \vee a) \vee b$

$= b' \vee a \vee b$
$= b' \vee b \vee a$
$= 1 \vee a$
$= 1$
Finally, $\left(a^{'} \wedge b^{'}\right) \wedge (a \vee b)$
$= a^{'} \wedge \left(b^{'} \wedge (a \vee b)\right)$
$= a^{'} \wedge \left(\left(b^{'} \wedge a\right) \vee \left(b^{'} \wedge b\right)\right)$
$= a^{'} \wedge \left(\left(b^{'} \wedge a\right) \vee 0\right)$
$= a^{'} \wedge \left(b^{'} \wedge a\right)$
$= a^{'} \wedge b^{'} \wedge a$
$= a^{'} \wedge a \wedge b^{'}$
$= 0 \wedge b^{'} = 0$ ■

Definition 150 *A* ***Boolean Lattice*** *is a bounded distributive lattice in which every element has a complement.*

Definition 151 *A* ***Boolean ring*** *B is a ring with identity in which $x^2 = x$ for all $x \in B$*

Exercise 152 *A Boolean ring B is commutative and has a characteristic of 2*

Solution 153 $2x = x + x = (x + x)^2 = 2x^2 + 2x^2 = 2x + 2x$
Hence, $2x = 0\ \forall x \in B\backslash \{0\}$
Therefore, $char(B) = 2$
$x + y = (x + y)^2 = x^2 + y^2 + xy + yx = x + y + xy + yx$
Hence, $xy + yx = 0$. *Since* $2xy = 0$,
or, $xy = xy \implies xy + 0 = xy \implies xy + xy + yx = xy \implies 2xy + yx = xy$ *or* $xy = yx$

Exercise 154 *If B is a Boolean ring, then B, partially ordered by $x \leq y \iff xy = x$ is a Boolean Lattice $\mathcal{L} = (B, ., +)$ in which where $x.y = x \wedge y$ and $x \vee y = x + y - xy$ and $x^{'} = 1 - x$*

Solution 155 *Trivially, $x^2 = x$ hence $\wedge$ is idempotent. Also, the both the operations are commutative since a Boolean ring is commutative in both operations. Multiplication is, by default, associative. To show that join is associative,* $x \vee (y \vee z) = x \vee (y + z - yz) = x + (y + z - yz) + x(y + z - yz) = x + y + z - yz + xy + xz - xyz$

The other law equals $(x \vee y) \vee z = (x + y - xy) \vee z = x + y - xy + z + (x + y - xy) z$

The sides can be shown to be equal by recalling the fact that the characterisitic of this ring is 2.

For the absorption laws, $(x \vee y) \wedge y = (x + y - xy) y = xy + y^2 - xy = y^2 = y$ *and* $(x \wedge y) \vee y = (xy) \vee y = xy + y - xy^2 = xy + y - xy = y$

Exercise 156 *Let D be the set of all positive divsors of some $n \in \mathbb{N}$, partially ordered by $x \leq y$ if and only if $x \mid y$. Show that D is distributive lattice. When is D a Boolean lattice?*

Solution 157 *We will make do with the usual conversion of order to meet and join. i.e. $x \wedge y = y \iff y \leq x$ and its dual. This does indeed form a lattice, as already proved. We can define meet and join by $x \wedge y = \gcd(x, y)$ and $x \vee y = \operatorname{lcm}(x, y)$. If we write $x = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, $y = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k}$ where the p_i are distinct primes and the α_i and β_i are non-negative integers, the $\gcd(x, y) = \prod\limits_{1 \leq i \leq k} p_i^{\min(\alpha_i, \beta_i)}$, $\operatorname{lcm}(x, y) = \prod\limits_{1 \leq i \leq k} p_i^{\max(\alpha_i, \beta_i)}$. Hence if $z = p_1^{c_1} p_2^{c_2} ... p_k^{c_k}$, c_i non-negative integral, then $x \vee (y \wedge z) = \operatorname{lcm}(x, \gcd(y, z)) = \prod\limits_{1 \leq i \leq k} p_i^{\max(\alpha_i, \min(\beta_i, c_i))}$ and $\gcd(\operatorname{lcm}(x, y), \operatorname{lcm}(x, z)) = \prod\limits_{1 \leq i \leq k} p_i^{\min(\max(\alpha_i, \beta_i), \max(\alpha_i, c_i))}$. Now the set of non-negative integers with the natural order is totally ordered and $\max(\alpha_i, \beta_i) = \alpha_i \vee \beta_i$ and $\min(\alpha_i, \beta_i) = \alpha_i \wedge \beta_i$ in this lattice. Hence, the distributive law in this lattice leads to the relation $\max(\alpha_i, \min(\alpha_i, c_i)) = \min(\max(\alpha_i, \beta_i), \max(\alpha_i, c_i))$. Then we have $\operatorname{lcm}(x, \gcd(y, z)) = \gcd(\operatorname{lcm}(x, y), \operatorname{lcm}(x, z))$*

Exercise 158 *A cofinite subset of a set X is a subset S of X whose complement $X - S$ is finite. Show that the subsets of X that are either finite or co-finite constitute a Boolean lattice*

Solution 159 *Of course the collection of subsets forms a lattice as has been already proven. The distributive and De Morgan's laws are well-known. The empty set is finite so that does with the least element. To prove that the largest element X is a member of this, we only need to observe that $X - X$ is finite.*

A central idempotent of a ring R with identity is an element e of R such that $e^2 = e$ and $ex = xe$ for all x. Show that the central idempotents of R constitute a Boolean lattice when ordered by $e \leq f \iff ef = e$

Solution 160 *Since we have an idempotent ring with identity, we have commutativy as was proved above*

Define $e \vee f = e + f - ef$ and $e \wedge f = ef$.
Now, for idempotentcy,
$e \vee e = e + e - e^2 = e$
$e \wedge e = e^2 = e$.
For commutativity,
$e \vee f = e + f - ef = f + e - fe = f \vee e$
$e \wedge f = ef = fe = f \wedge e$
For associativity,
$(e \vee f) \vee g = (e + f - ef) \vee g$
$= e + f - ef + g - g(e + f - ef)$
$= e + f - ef + g - ge - gf + gef$
$= e + f - ef + g - ge - gf + gef - 2gef$

$= e + f + g - gf - gef - ef - eg$
$= e + (f + g - gf) - e(f + g + gf)$
$= e + (f + g - gf) - e(f + g + gf - 2gf)$
$= e + (f + g - gf) - e(f + g - gf)$
$= e \vee (f + g - fg) = e \vee (g \vee f)$
and $e \wedge (f \wedge g) = e \wedge (fg) = e(fg) = (ef)g$
$= (e \wedge f)g = (e \wedge f) \wedge g$
For the absorption laws,
$(e \vee f) \wedge e = (e + f - ef)e = e^2 + fe - efe$
$= e + fe - e^2 f = e + fe - ef = e$
and also $(e \wedge f) \vee e = (ef) + e - (ef)e = ef + e - e^2 f = efef + e - ef = e$
For distributive laws,
$(e \wedge f) \vee g = (ef) \vee g = ef + g - efg =$
$= ef + eg - eg + gf - gf - 2efg + 2efg + g - efg$
$= ef + eg - efg + gf + g - fg - efg - eg + efg$
$= ef + eg - efg + gf + g^2 - fg^2 - efg - eg^2 + efg^2$
$= (e + g - eg)(f + g - fg)$
$= (e \vee g) \wedge (g \vee f)$
Similarly, $(e \vee f) \wedge g$
$= (e + f - ef)g$
$= eg + fg - efg$
$= eg + fg - e^2 fg$
$= eg + ef - efeg$
$= (eg) \vee (ef)$
$= (e \wedge g) \vee (e \wedge f)$
We use the additive identity 0 *and the multiplicative identity* 1 *as our bounds*
To this end, we see that $e \wedge 1 = (e)(1) = e$, $e \vee 1 = e + 1 - (e)(1) = 1$
Furthermore, $e \wedge 0 = (e)(0) = 0$ *and* $e \vee 0 = e + 0 - (e)(0)e$
Clearly, both bounds are idempotent and central
Finally, if we define the our complements as $e^c = 1 - e$, *then* $e \vee e^c$
$= e - (1 - e) - (e)(1 - e)$
$= e - 1 + e - e + e^2 = 2e$
$= -1 = -1 + 2(1) = 1$

Furthermore, $e \wedge e^c = (e)(1 - e) = e - e^2 = e - e = 0$. *A final test is the satisfaction of the De Morgan Laws.* $(e \vee f)^c = (e + f - ef)^c = 1 - e - f + ef$
$1 - e + f(e - 1) = -1(e - 1) + f(e - 1)$
$(e - 1)(f - 1) = (1 - e)(1 - f) = e^c \wedge f^c$
Also, $(e \wedge f)^c = (ef)^c = 1 - ef$
$= 1 - ef + f - f + 2 + e - e$
$= 1 - e + 1 - f - 1 + f + e - ef$
$= (1 - e) + (1 - f) - (1 - e)(1 - f)$
$= e^c \vee f^c$

2 Fuzzy Theory

This portion of the lecture notes is majorly copied from Xuzhu Wang, Da Ruan and Etienne E. Kerre's *Mathematics of Fuzziness – Basic Issues.*

2.1 Fuzzy set theory

In this chapter, we focus on the introduction of fundamentals in fuzzy set theory, including some set-theoretic operations and their extensions, the decomposition of a fuzzy set, and mathematical representations of fuzzy sets in terms of a nest of sets. Towards the end of the chapter, fuzzy sets taking values in $[0, 1]$ are extended to those on a lattice and a similar investigation is carried out.

According to Cantor, a set consists of some elements which are definite. In other words, for a given element, whether it belongs to the set or not should be clear. As a consequence, a set can only be employed to describe a concept which is crisply defined. For example, a collection of cities with the population more than 5 millions forms a set since we can judge that a given city is in this set or not without vagueness. In traditional mathematics, all the involved concepts ranging all the way from the complex numbers and matrices to geometric transformations and algebraic structures are in this category. However, in the real world, mankind often uses concepts which are quite vague. For example, we say that a man is young or middle-aged, an object is expensive or cheap, a tomato is red and mature, a number is large or small, a car is slow or fast and so on. Let us take young as an illustration.

Suppose A is a 20-year-old man. Maybe you think A is certainly young. Now comes a man B only one day older than A. Of course, B is still young. Then how about a man only one day older than B? Continuing in this way, you will find it difficult to determine an exact age beyond which a man will be middle-aged. As a matter of fact, there is no sharp line between young and middle-aged. The transition from one concept to the other is gradual. This gradualness results in the vagueness of the concept young, which in return makes the boundary of the set of all young men unclear.

In 1965, Zadeh introduced the concept of fuzzy sets just in order to represent this class of sets. Zadeh assigns a number to every element in the universe, which indicates the degree (grade) to which the element belongs to a fuzzy set. In this interpretation, everybody has a degree to which he/she is young (eventually the degree may be 0 or 1). The people with different ages may have different degrees. To formulate this concept of fuzzy set mathematically, we present the following definition.

Definition 161 *Let X be the universe. A mapping $A : X \longrightarrow [0, 1]$ is called a **fuzzy set** on X. The value $A(x)$ of A at $x \in X$ stands for the degree of membership of x in A.*

The set of all fuzzy sets on X will be denoted by $F(X)$. $A(x) = 1$ means full membership, $A(x) = 0$ means non-membership and intermediate values between

0 and 1 mean partial membership. $A(x)$ is referred to as a membership function as x varies in X.

Theorem 162 *Let X be a non-empty subset. Then, there exists an isomorphism between $(\mathcal{P}(X), \cap, \cup, c)$ and $(Ch(X), \vee, \wedge, c)$ where $\mathcal{P}(X)$ is the powerset of X and $Ch(X)$ is the set of two-valued charactersitic functions on X.*

Proof. Let $\chi_S : X \longrightarrow \{0,1\}$ be a characteristic function for a subset S in $Ch(X)$. Thus, for each element of $\mathcal{P}(X)$, we have an element of $Ch(X)$. Now, let $f : \mathcal{P}(X) \longrightarrow Ch(X)$ be a mapping such that $f(A) = \chi_A$.

We will first prove that f is bijective. Clearly, f is onto since for each characteristic function, we can construct a corresponding set. Let A and B be two sets such that $f(A) = f(B)$. Take $x \in A$. Then, $\chi_A(x) = 1 = \chi_B(x) \implies x \in B$. Similarly, $B \subseteq A$.

To prove that the structures are preserved, $f(A \cup B) = \chi_{A \cup B} = \chi_A \vee \chi_B = f(A) \vee f(B)$

Next, $f(A \cap B) = \chi_{A \cup B} = \chi_A \wedge \chi_B = f(A) \wedge f(B)$

Finally, $f(A^c) = \chi_{A^c} = 1 - \chi_A = 1 - f(A) = f(A)^c$ ■

It follows from the isomorphism between $(\mathcal{P}(X), \cap, \cup, ^c)$ and $(Ch(X), \vee, \wedge, ^c)$ that every subset of X may be regarded as a mapping from X to $\{0,1\}$. In this sense an ordinary set is also a fuzzy set, whose membership function is just its characteristic function. Accordingly we shall identify the membership degree $A(x)$ with the value $\chi_A(x)$ of the characteristic function χ_A at x when A is an ordinary set. For the two extreme cases $\varnothing$ (the empty set) and X (the entire set), the membership functions are defined by $\forall x \in X$, $\varnothing(x) = 0$ and $X(x) = 1$, respectively. In contrast with fuzzy sets, ordinary sets are sometimes termed by crisp sets in this book.

Example 163 *Let O denote old and Y denote young. We limit the scope of age to $X = [0, 100]$. Then both O and Y are fuzzy sets that are respectively defined by Zadeh as follows:*

$$O(x) = \begin{cases} \left[1 + \left(\frac{x-50}{5}\right)^{-2}\right]^{-1} & \textit{if } 50 \leq x \leq 100 \\ 0 & \textit{otherwise} \end{cases}$$

and

$$Y(x) = \begin{cases} \left[1 + \left(\frac{x-25}{5}\right)^{2}\right]^{-1} & \text{if } 25 \leq x \leq 100 \\ 1 & \text{otherwise} \end{cases}$$

For instance, $O(60) = 0.8$ and $Y(30) = 0.5$.

Example 164 *As known to us, all the involved quantities are precise in traditional mathematics. With fuzzy sets, we can model the so-called fuzzy data. For instance, the fuzzy datum $A =$ "around 1" may be represented by: $\forall x \in R$,*

$$A(x)=\begin{cases} x & 0\le x\le 1 \\ 2-x & 0\le x\le 2 \\ 0 & \text{otherwise}\end{cases}$$

In the case of infinite universe, a fuzzy set may be represented by its membership function as in the above example. If the universe is finite, say, $X=\{x_1,x_2,...,x_n\}$, the fuzzy set A on X is represented by $A=A(x_1)/x_1+A(x_2)/x_2+...+A(x_n)/x_n$.

For example, the fuzzy set $S=$ *several* on $X=\{1,2,\dots,10\}$ may be written as:

$S=0/1+0.6/2+1/3+1/4+1/5+0.9/6+0.8/7+0.7/8+0.6/9+0/10$.

For the sake of conciseness, the terms with degree 0, e.g. the terms 0/1, 0/10 in S, are dropped. As a result,

$S=0.6/2+1/3+1/4+1/5+0.9/6+0.8/7+0.7/8+0.6/9$. Importantly, the choice of a membership function is context-dependent. It is clearly different that the temperature of a steel-smelting furnace is high and the temperature of a human body is high. Even in a same context, the choice is dependent on the observer. It is certainly different from Zadeh's if you form the membership function of the fuzzy concept young.

Next we introduce some set-theoretic operations of fuzzy sets formulated by Zadeh. Let A and B be two fuzzy sets on X. The union $A\cup B$ of A and B is defined by $\forall x\in X$, $(A\cup B)(x)=\max(A(x),B(x))$ (or simply $A(x)\vee B(x)$);

The intersection $A\cap B$ of A and B is defined by $\forall x\in X$, $(A\cap B)(x)=\min(A(x),B(x))$ (or simply $A(x)\wedge B(x)$);

The complement A^c of A is defined by $\forall x\in X$, $A^c(x)=1-A(x)$.

Remark 165 *As in crisp case, the union (intersection) of fuzzy sets A and B represents "A or (resp. and) B", and the complement of A means "not A".*

Example 166 *Let* $X=\{1,2,\dots,10\}$. $A=$*small*$=1/1+0.8/2+0.6/3+0.4/4+0.2/5$, $B=$*large*$=0.2/4+0.4/5+0.6/6+0.8/7+1/8+1/9+1/10$.

Then, not small
$=A^c=0.2/2+0.4/3+0.6/4+0.8/5+1/6+1/7+1/8+1/9+1/10$,
not large
$=B^c=1/1+1/2+1/3+0.8/4+0.6/5+0.4/6+0.2/7$,
not small and not large
$=A^c\cap B^c=0.2/2+0.4/3+0.6/4+0.6/5+0.4/6+0.2/7$.

Exercise 167 *Assume two fuzzy sets* A_1 *and* A_2 *on* $X=\{x_1,x_2,x_3,x_4\}$ *are defined by* $A_1=0.1/x_1+0.9/x_2+0.6/x_3$, $A_2=0.9/x_1+0.7/x_2+0.6/x_3+0.8/x_4$. *Find* A_1^c, $A_1\cup A_2$ *and* $A_1\cap A_2$.

Solution 168 $A_1^c=0.1/x_1+0.3/x_2+0.4/x_3+0.8/x_4$
$A_1\cup A_2=0.9/x_1+0.9/x_2+0.6/x_3+0.8/x_4$
$A_1\cap A_2=0.1/x_1+0.7/x_2+0.6/x_3$

Definition 169 *If $\forall x \in X$, $A(x) \leq B(x)$, then we call that A is a* ***subset*** *of B or A is contained in B, denoted by $A \subseteq B$. If $\forall x \in X$, $A(x) = B(x)$, then A and B are called* ***equal****, denoted by $A = B$. Obviously, $A = B$ iff $A \subseteq B$ and $B \subseteq A$. If $A \neq \varnothing$, $A \subseteq B$ and $\exists x \in X$ such that $A(x) < B(x)$, then we say that A is* ***properly contained*** *in B, denoted by $A \subset B$.*

It follows immediately from the definitions that

Theorem 170 $\forall A, B, C, D \in F(X)$,
(1) $A \cap B \subseteq A$ and $A \subseteq A \cup B$;
(2) $A \subseteq B \iff A \cup B = B \iff A \cap B = A$;
(3) $A \subseteq B$ and $C \subseteq D \implies A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$;
(4) $A \subseteq B \implies B^c \subseteq A^c$.

Proof. (1) $(A \cap B)(x) = \min(A(x), B(x)) \leq A(x)$. Since this is valid for all x, therefore $A \cap B \subseteq A$

Similarly, $A(x) \leq \max(A(x), B(x)) \implies A \subseteq A \cup B$

(2) $A \subseteq B \iff \forall x \in X, A(x) \leq B(x) \iff \forall x \in X, \max\{B(x), A(x)\} = B(x) \iff A \cup B = B$

and $A \subseteq B \iff \forall x \in X, A(x) \leq B(x) \iff \forall x \in X, \min\{B(x), A(x)\} = A(x) \iff A \cap B = B$

(3) $A \subseteq B$ and $C \subseteq D \implies \forall x \in X, A(x) \leq B(x)$ and $\forall x \in X, C(x) \leq D(x)$ from which we have $\max(A(x), C(x)) \leq \max(B(x), D(x))$ and $\min(A(x), C(x)) \leq \min(B(x), D(x))$

(4) $A \subseteq B \implies \forall x \in X, A(x) \leq B(x) \implies \forall x \in X,\ 1 - B(x) \leq 1 - A(x) \implies B^c \subseteq A^c$ ■

In addition, we have the following important conclusion concerning the fuzzy set-theoretic operations.

Theorem 171 $(F(X), \cup, \cap, ^c)$ *is a soft algebra, i.e. $F(X)$ satisfies: $\forall A, B, C \in F(X)$,*
(1) idempotency: $A \cup A = A$, $A \cap A = A$;
(2) commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$;
(3) associativity: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$;
(4) absorption laws: $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$;
(5) distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
(6) the existence of the greatest and least element: $\varnothing \subseteq A \subseteq X$.
(7) involution: $(A^c)^c = A$;
(8) De Morgan laws: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Proof. In each proof, the arguments are valid $\forall x \in X$, so that the function notation can be justified in terms of functions.

(1) $(A \cup A)(x)$
$= \max\{A(x), A(x)\}$
$= \max\{A(x)\} = A(x)$
(2) $(A \cup B)(x)$

$= \max\{A(x), B(x)\}$
$= \max\{B(x), A(x)\} = (B \cup A)(x)$
and $(A \cap B)(x)$
$= \min\{A(x), B(x)\}$
$= \min\{B(x), A(x)\}$
$= (B \cap A)(x)$
(3) $((A \cup B) \cup C)(x)$
$= \max\{\max\{A(x), B(x)\}, C(x)\}$
$= \max\{A(x), B(x), C(x)\}$
$= \max\{A(x), \max\{B(x), C(x)\}\}$
$= (A \cup (B \cup C))(x)$
(4) $(A \cup (A \cap B))(x)$
$= \max\{\min\{A(x), B(x)\}, A(x)\}$

Assume $\min\{A(x), B(x)\} = B(x)$. Then, $\max\{A(x), B(x)\} = A(x)$. On the other hand, if $\min\{A(x), B(x)\} = A(x)$, then $\max\{A(x), A(x)\} = A(x)$. Clearly, these are the only two possibilities.

$$(A \cap (A \cup B))(x) = \min\{\max\{A(x), B(x)\}, A(x)\}$$

Assume $\max\{A(x), B(x)\} = A(x)$. Then, the result is true again by idempotency. If $\max\{A(x), B(x)\} = B(x)$, then $\min\{B(x), A(x)\} = A(x)$

(5) Since left distributive law implies the right distributive law, we will only prove one namely $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$(A \cup (B \cap C))(x) = \max\{A(x), \min\{B(x), C(x)\}\}$. Now, since we have real numbers $A(x)$, $B(x)$ and $C(x)$, therefore

$$\min\{B(x), C(x)\} = \begin{cases} C(x) & \text{if } B(x) \geq C(x) \\ B(x) & \text{if } C(x) > B(x) \end{cases}$$

and similarly for maximum we have

$$\max\{A(x), Y(x)\} = \begin{cases} A(x) & \text{if } A(x) \geq Y(x) \\ Y(x) & \text{if } Y(x) > A(x) \end{cases}$$

where $Y(x) = \min\{B(x), C(x)\}$. Combining,

$$\max\{A(x), \min\{B(x), C(x)\}\} = \begin{cases} A(x) & \text{if } A(x) \geq \min\{B(x), C(x)\} \\ \min\{B(x), C(x)\} & \text{if } \min\{B(x), C(x)\} > A(x) \end{cases}$$

It is easy to see that maximum will thus distribute over minimum by arguing on a case-by-case basis (I'm too lazy to; don't want to get my hands dirty) and using the Law of Trichotomy to show that $\max\{A(x), \min\{B(x), C(x)\}\} = \min\{\max\{A(x), B(x)\}, \max\{A(x), C(x)\}\}$

Therefore,

$$\max\{A(x), \min\{B(x), C(x)\}\} = \min\{\max\{A(x), B(x)\}, \max\{A(x), C(x)\}\}$$

$= (A \cup B)(x) \cap (A \cup C)(x)$

(6) By definition, $\forall x \in X$, $\varnothing(x) = 0$ and $X(x) = 1$.
Since for any A, $A(x) \in [0, 1]$, therefore $\varnothing(x) \leq A(x) \leq X(x)$
$\Longrightarrow \varnothing \subseteq A \subseteq X$
(7) $(A^c)^c(x)$
$= 1 - (1 - A(x)) = A(x)$
$\Longrightarrow (A^c)^c = A$
(8) $(A \cup B)^c(x) = 1 - (A \cup B)(x)$
$= 1 - \max\{A(x), B(x)\}$
If $A(x) \leq B(x)$, then $1 - B(x) \leq 1 - A(x)$ and
$\min\{1 - B(x), 1 - A(x)\} = 1 - B(x) = 1 - \max\{A(x), B(x)\}$
$= \min\{1 - B(x), 1 - A(x)\}$
If $B(x) \leq A(x)$, then $1 - A(x) \leq 1 - B(x)$ and
$\min\{1 - B(x), 1 - A(x)\} = 1 - A(x) = 1 - \max\{A(x), B(x)\}$
$= \min\{1 - B(x), 1 - A(x)\}$
Therefore, in either case we have

$$1 - \max\{A(x), B(x)\} = \min\{1 - A(x), 1 - B(x)\}$$

$= \min\{A^c(x), B^c(x)\}$
$= (A^c \cap B^c)(x)$
$\Longrightarrow (A \cup B)^c = (A^c \cap B^c)$
For the second, $(A \cap B)^c(x)$
$= 1 - (A \cap B)(x)$
$= 1 - \min\{A(x), B(x)\}$
If $A(x) \leq B(x)$, then $1 - B(x) \leq 1 - A(x)$ and
$\max\{1 - B(x), 1 - A(x)\} = 1 - A(x) = 1 - \min\{A(x), B(x)\}$
If $B(x) \leq A(x)$, then $1 - A(x) \leq 1 - B(x)$ and
$\max\{1 - B(x), 1 - A(x)\} = 1 - B(x) = 1 - \min\{A(x), B(x)\}$
Therefore, $1 - \min\{A(x), B(x)\} = \max\{1 - B(x), 1 - A(x)\}$
$= (A^c \cup B^c)(x)$
$\Longrightarrow (A \cap B)^c = (A^c \cup B^c)$ ■

From the above proof, we see that properties of $(F(X), \cup, \cap, ^c)$ are largely dependent on properties of $([0, 1], \max, \min, ^c) = ([0, 1], \vee, \wedge, ^c)$ since the set-theoretic operations are defined pointwise. In this sense, $[0, 1]$ is regarded as the underlying structure set of $F(X)$. As a result, it is not strange that $(F(X), \cup, \cap, ^c)$ has the same algebraic structure as $([0, 1], \vee, \wedge, ^c)$. The partial order relation $\leq$ in the soft algebra $(F(X), \cup, \cap, ^c)$ is $\subseteq$.

Proof. In the proof, again, the argument is valid for any $x \in X$. Clearly, for any $A \in F(X)$, $A(x) \leq A(x)$ so that $A \subseteq A$, making $\subseteq$ reflexive. Next, if $A \subseteq B$ and $B \subseteq A$, then $A(x) \leq B(x)$ and $B(x) \leq A(x)$ so that $A(x) = B(x)$. Finally, if $A \subseteq B$ and $B \subseteq C$, then $A(x) \leq B(x)$ and $B(x) \leq C(x)$ $\Longrightarrow A(x) \leq C(x) \Longrightarrow A \subseteq C$, making $\subseteq$ a bonafide partial order. ■

Like $([0, 1], \vee, \wedge, ^c)$, $(F(X), \cup, \cap, ^c)$ is not a Boolean algebra since it is not complemented, i.e. $A \cap A^c = \varnothing$ and $A \cup A^c = X$ do not hold generally. To illustrate this point, consider the fuzzy set A defined by $\forall x \in X$, $A(x) = 0.5$. Then $\forall x \in X$, $(A \cap A^c)(x) = (A \cup A^c)(x) = 0.5$ while $\varnothing(x) = 0$ and

$X(x) = 1$. Consequently, $A \cap A^c \neq \varnothing$ and $A \cup A^c \neq X$, which indicates that neither the law of contradiction nor the law of excluded middle hold. It is quite natural considering that these two laws are the logical foundation of traditional mathematics. In this sense, the emergence of fuzzy sets gives birth to a completely new logic – fuzzy logic, and hence to a completely new mathematics – mathematics of fuzziness.

Exercise 172 *If $A, B, C \in F(X)$, show that (1) $\{A \cap [(B \cap C) \cup (A^c \cap C^c)]\} \cup C^c = (A \cap B \cap C) \cup C^c$ and (2) $(A \cap B) \cup (B \cap C) \cup (A \cap C) = (A \cup B) \cap (B \cup C) \cap (A \cup C)$*

Solution 173 $\{A \cap [(B \cap C) \cup (A^c \cap C^c)]\} \cup C^c$
$= (A \cap (B \cap C)) \cup (A \cap (A^c \cap C^c)) \cup C^c$
$= (A \cap B \cap C) \cup (A \cap A^c \cap C^c) \cup C^c$
$= (A \cap B \cap C) \cup [(A \cup C^c) \cap (C^c \cup (A^c \cap C^c))]$
$= (A \cap B \cap C) \cup [(A \cup C^c) \cap C^c]$
$= (A \cap B \cap C) \cup C^c$
(2) $(A \cap B) \cup (B \cap C) \cup (A \cap C)$
$=$

Exercise 174 *The difference $A - B$ and symmetric difference $A \vartriangle B$ of two fuzzy sets A and B are respectively defined by $A - B = A \cap B^c$ and $A \vartriangle B = (A - B) \cup (B - A)$. (i) Use $A(x)$ and $B(x)$ to express $(A - B)(x)$ and $(A \vartriangle B)(x)$. (ii) Assume A and B are two fuzzy sets on $X = \{a, b, c, d, e, f, g\}$ defined by $A = 0.5/b + 0.4/c + 1/d + 0.7/f$, $B = 0.3/a + 0.9/b + 0.4/c + 1/d + 0.6/e + 1/g$. Find $A - B$ and $A \vartriangle B$. (iii) Show that $(A \vartriangle B) \vartriangle C = A \vartriangle (B \vartriangle C)$.*

Solution 175 *(i)* $(A - B)(x) = (A \cap B^c)(x) = \min\{A(x), 1 - B(x)\}$ *and* $(A \vartriangle B)(x) = \max\{(A - B)(x), (B - A)(x)\}$
$= \max\{\min\{A(x), 1 - B(x)\}, \min\{B(x), 1 - A(x)\}\}$
(ii) $A - B = 0.1/b + 0.4/c + 0.7/f$ *and* $A \vartriangle B$ *can be found using the formula above*
(iii) $((A \vartriangle B) \vartriangle C)$
$= \{[(A - B) \cup (B - A)] - C\} \cup \{C - [(A - B) \cup (B - A)]\}$
$= \{[(A \cap B^c) \cup (B \cap A^c)] \cap C^c\} \cup \{C \cap [(A \cap B^c) \cup (B \cap A^c)]^c\}$
$= \{[((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup A^c)] \cap C^c\} \cup \{C^c \cup [(A^c \cup B) \cap (B^c \cup A)]\}$
$= \{(B \cup A) \cap (B^c \cup B) \cap (A \cup A) \cap (B^c \cup A^c) \cap C^c\} \cup \{C^c \cup [((A^c \cup B) \cap B^c) \cup ((A^c \cup B) \cap A)]\}$
$= \{(B \cup A) \cap (B^c \cup B) \cap A \cap (B^c \cup A^c) \cap C^c\} \cup C^c \cup (A^c \cap B^c) \cup (B \cap B^c) \cup (A^c \cap A) \cup (B \cap A)$
some magic
$= \{A \cap (B^c \cup C) \cap (C^c \cup B)\} \cup \{[(B \cap C^c) \cup (C \cap B^c)] \cap A^c\}$
$= \{A \cap [(B \cap C^c) \cup (C \cap B^c)]^c\} \cup \{[(B \cap C^c) \cup (C \cap B^c)] \cap A^c\}$
$= \{A - [(B - C) \cup (C - B)]\} \cup \{[(B - C) \cup (C - B)] - A\}$
$= (A \vartriangle (B \vartriangle C))$

The union and intersection operations can be extended as follows: For $A_i \in F(X)$ where $i \in I$, an arbitrary index set, $\bigcup_{i\in I} A_i(x) = \left(\bigcup_{i\in I} A_i\right)(x) = \sup\{A_i(x) \mid i \in I\} = \left(\bigvee_{i\in I} A_i\right)(x) = \bigvee_{i\in I} A_i(x)$ and similarly $\bigcap_{i\in I} A_i(x) = \left(\bigcap_{i\in I} A_i\right)(x) = \inf\{A_i(x) \mid i \in I\} = \left(\bigwedge_{i\in I} A_i\right)(x) = \bigwedge_{i\in I} A_i(x)$

Definition 176 *The set $\{x \mid A(x) = 1\}$ is said to be the **kernel** of A, denoted by $\ker(A)$;*

Definition 177 *The set $\{x \mid A(x) > 0\}$ is called the **support** of A, denoted by $supp(A)$;*

Definition 178 *The number $\bigvee_{x\in X} A(x)$ is called the **height** of A, denoted by $hgt(A)$*

Definition 179 *The number $\bigwedge_{x\in X} A(x)$ is referred to as the **plinth** of A, denoted by $plt(A)$.*

Definition 180 *If $\ker(A) = \varnothing$, then A is called a **normal fuzzy set**.*

For the characteristic function χ_S pertaining to a subset S of X, we know that $\forall x, \chi_{A\cap B}(x) = \min(\chi_A(x), \chi_B(x))$ which justifies Zadeh's use of the minimum operator in formulating the intersection of two fuzzy sets. It is also seen that $\chi_{A\cap B}(x) = \max(\chi_A(x)+\chi_B(x)-1, 0)$. Hence it is also reasonable to define the intersection of fuzzy sets $A, B \in F(X)$ by $(A\cap B)(x) = A(x)B(x)$ or by $(A\cap B)(x) = \max\{A(x) + B(x) - 1, 0\}$ if we consider the intersection of fuzzy sets as an extension of the intersection of crisp sets. The similar argument exists for the definition of the complement and union. In other words, to extend operations of crisp sets to the fuzzy case, there may be multiple alternative ways. The definition in the previous section is just one of them. More generally, the operation of intersection, union and complement can be formulated by means of the so-called t-norms, t-conorms and fuzzy negations, respectively, together with fuzzy implications and fuzzy equivalencies.

Definition 181 *If $\eta : [0,1] \longrightarrow [0,1]$ is decreasing and satisfies the boundary conditions $\eta(0) = 1$ and $\eta(1) = 1$, then η is called a (fuzzy) **negation**.*

If we define η by $\forall x \in [0,1]$, $\eta(x) = 1 - x$,then η is a negation, which is called the standard negation.

Example 182 *The mapping $\eta_i : [0,1] \longrightarrow [0,1]$ defined by $\forall x \in [0,1]$,*

$$\eta_i(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$$

is a negation, which is called the intuitionistic negation; and

$$\eta_{d_i}(x) = \begin{cases} 0 & x = 1 \\ 1 & x < 1 \end{cases}$$

is also a negation, which is called the dual intuitionistic negation.

Proposition 183 $\eta_i(x) \leq \eta(x) \leq \eta_{d_i}(x)$

Proof. If $x = 0$ and $x = 1$, the condition is trivially satisfied. Assume $x \neq 0$ and $x \neq 1$. Since $0 \leq \eta(x) \leq 1$, therefore $\eta_i(x) \leq \eta(x) \leq \eta_{d_i}(x)$ because $\eta_{d_i}(x) = 1$ for $x \neq 1$ and $\eta_i(x) = 0$ for $x \neq 0$ ■

Definition 184 *A strictly decreasing continuous negation is called a* ***strict negation****. A strict negation η is called a strong negation if it satisfies the involution: $\forall x \in [0,1]$, $\eta(\eta(x)) = x$.*

It follows that the intuitionistic and dual intuitionistic negation is not strict since for $x < y < 1$, $\eta_{d_i}(y) \leq \eta_{d_i}(x)$ and for $0 < x < y$, $\eta_i(y) \leq \eta_i(x)$ because then $\eta_i(y) = \eta_i(x) = 1$ and $\eta_{d_i}(y) = \eta_{d_i}(x) = 0$

Example 185 *The function $\eta : [0,1] \longrightarrow [0,1]$ such that $\eta(x) = 1 - x^2$ is a strict, non-strong negation*

Proof. $\eta(0) = 1 - 0^2 = 1$ and $\eta(1) = 1 - 1^2 = 0$. Next, if $0 < x < y < 1$, then $y^2 < x^2$ and $1 - y^2 < 1 - x^2$ which implies $\eta(y) < \eta(x)$ but $\eta(\eta(x)) = 1 - \eta(x) = 1 - \left(1 - x^2\right) = x^2$ ■

Definition 186 *Let $\phi : [a,b] \longrightarrow [a,b]$ be a strictly increasing and continuous function. If ϕ satisfies $\phi(a) = a$ and $\phi(b) = b$, then ϕ is called an* ***automorphism*** *on $[a,b]$.*

Example 187 *$\phi_1(x) = x$ is an automorphism on $[a,b]$.*

Example 188 *$\phi_2(x) = x^2$ is an automorphism on $[0,1]$.*

Example 189 *$\phi_3(x) = x^2 + x - 1/4$ is an automorphism on $[-1/2, 1/2]$ because $\phi_3(x)$ is a quadratic polynomial, making it continuous and $\phi_3(-1/2) = 1/4 - 1/2 - 1/4 = -1/2$ whereas on the other hand $\phi_3(1/2) = 1/4 + 1/2 - 1/4 = 1/2$*

Lemma 190 *Let η_1 and η_2 be two strict negations. Then there exist two automorphisms ϕ and ψ on $[0,1]$ such that $\eta_2 = \psi \circ \eta_1 \circ \phi$*

Proof. This proof will construct two such automorphisms. Since η_1 and η_2 are two continuous self-maps, there must exist fixed points. Let $s_1, s_2 \in [0,1]$ be such two fixed points. That is, $\eta_1(s_1) = s_1$ and $\eta_2(s_2) = s_2$.Since $\eta_1(0) = 1$ and $\eta_2(0) = 1$, we have $s_1 \neq 0$ and $s_2 \neq 0$. Let $t = s_2/s_1$. Define $\phi : [0,1] \longrightarrow [0,1]$ and $\psi : [0,1] \longrightarrow [0,1]$ as follows:

$$\phi(x) = \begin{cases} \frac{x}{t} & x \leq s_2 \\ \eta_1^{-1}\left(\frac{\eta_2(x)}{t}\right) & x > s_2 \end{cases}$$

and

$$\psi(x) = \begin{cases} tx & x \leq s_1 \\ \eta_2\left[t\eta_1^{-1}(x)\right] & x > s_1 \end{cases}$$

We show that this definition of ϕ and ψ is an automorphism on $[0,1]$. If $x = 0$, then $x \leq s_2, s_1$ and $\phi(0) = 0$ and $\psi(0) = 0$. If $x = 1$, then $x > s_2, s_1$ and $\phi(1) = \eta_1^{-1}\left(\frac{\eta_2(1)}{t}\right) = \eta_1^{-1}(0) = 1$ and $\psi(1) = \eta_2\left[t\eta_1^{-1}(1)\right] = \eta_2(0) = 1$. To show that the functions are continuous, it suffices to show continuity at s_1 and s_2. $\lim\limits_{x^+ \to s_2} \phi(x) = s_2/t = s_1$ whereas $\lim\limits_{x^- \to s_2} \phi(x) = \lim\limits_{x^- \to s_2} \eta_1^{-1}\left(\frac{\eta_2(x)}{t}\right) = \eta_1^{-1}\left(\lim\limits_{x^- \to s_2} \frac{\eta_2(x)}{t}\right) = \eta_1^{-1}\left(\frac{\lim_{x^- \to s_2} \eta_2(x)}{t}\right) = \eta_1^{-1}(s_1) = s_1$. For the second function, $\lim\limits_{x^+ \to s_1} \psi(x) = ts_1 = s_2$ and $\lim\limits_{x^- \to s_1} \psi(x) = \lim\limits_{x^- \to s_1} \eta_2\left[t\eta_1^{-1}(x)\right] = \eta_2\left[\lim\limits_{x^- \to s_1} t\eta_1^{-1}(x)\right] = \eta_2\left[t\eta_1^{-1}\left(\lim\limits_{x^- \to s_1} x\right)\right] = \eta_2\left[t\eta_1^{-1}(s_1)\right] = \eta_2(ts_1) = \eta_2(s_2) = s_2$.

Now, when $x < s_2$, then $x/t < s_2/t = s_1$. That is, for $x/t < s_1$, $\eta_1(s_1) = s_1 < \eta_1(x/t)$ from which we can say that $\psi(\eta_1(\phi(x))) = \psi(\eta_1(x/t)) = \eta_2\left[t\eta_1^{-1}(x)\right] = \eta_2\left[t\eta_1^{-1}(\eta_1(x/t))\right] = \eta_2(xt/t) = \eta_2(x)$

On the other hand, when $x \geq s_2$, $\eta_2(x) \leq \eta_2(s_2) = s_2 = s_1 t$ and thus $\frac{\eta_2(x)}{t} \leq s_1$ from which we can say that $\psi(\eta_1(\phi(x))) = \psi\left(\eta_1\left(\eta_1^{-1}\left(\frac{\eta_2(x)}{t}\right)\right)\right) = \psi\left(\frac{\eta_2(x)}{t}\right) = \frac{\eta_2(x)}{t}t = \eta_2(x)$. Thus, in both cases, the identity holds. ■

Lemma 191 *Let η_1 be a strict negation and two automorphisms ϕ and ψ exist on $[0,1]$ such that $\eta_2 = \psi \circ \eta_1 \circ \phi$. Then, η_2 is a strict negation.*

Proof. $\eta_2(0) = \psi(\eta_1(\phi(0))) = \psi(\eta_1(0)) = \psi(1) = 1$

$\eta_2(1) = \psi(\eta_1(\phi(1))) = \psi(\eta_1(1)) = \psi(0) = 0$

η_2 is continuous since the composition of continuous functions is continuous.

Finally, for $x \leq y$, $\phi(x) \leq \phi(y)$. Since η_1 is strict, therefore for $\phi(x) \leq \phi(y)$, $\eta(\phi(y)) < \eta(\phi(x))$ and finally $\eta_2(y) = \psi(\eta_1(\phi(y))) \leq \psi(\eta_1(\phi(x))) = \eta_2(x)$ ■

Theorem 192 *The negation $\eta : [0,1] \longrightarrow [0,1]$ is strict iff there exists two automorphisms $\psi : [0,1] \longrightarrow [0,1]$ and $\phi : [0,1] \longrightarrow [0,1]$ such that $\eta(x) = \psi(1 - \phi(x))$*

Proof. The negation $\eta_s(x) = 1 - x$ is strict and therefore we can have $\psi(x)$ and $\phi(x)$ such that $\eta = \psi(\eta_s(\phi(x))) = \psi(1 - \phi(x))$

Conversely, assume that $\eta(x) = \psi(1 - \phi(x))$. Then, for $x < y$, we have $\phi(x) < \phi(y) \implies 1 - \phi(y) < 1 - \phi(x) \implies \eta(y) = \psi(1 - \phi(y)) < \psi(1 - \phi(x)) = \eta(x)$ implying that η is strictly decreasing. ■

Lemma 193 *Let N_1 and N_2 be two strong negations. Then there exists an automorphism ϕ on $[0,1]$ such that $N_2 = \phi^{-1} \circ N_1 \circ \phi$.*

Proof. The automorphism $\phi(x) = x$ does a perfect job for this but this identity map is trivial and uninteresting. Notice that in the above proof, $\psi(x)$ and $\phi(x)$ are inverses of each other. ∎

Theorem 194 *The mapping* $N : [0,1] \longrightarrow [0,1]$ *is a strong negation iff there exists automorphism* $\psi : [0,1] \longrightarrow [0,1]$ *such that* $N(x) = \psi^{-1}(1 - \psi(x))$

Proof. This is a direct consequence of using $N = N_2$ and $N_1 = 1 - x$ in the above lemma. Conversely, we start to show that N is a negation if $N(x) = \psi^{-1}(1 - \psi(x))$. $N(0) = \psi^{-1}(1 - \psi(0)) = \psi^{-1}(1 - 0) = \psi^{-1}(1) = 1$. Similarly, $N(1) = \psi^{-1}(1 - \psi(1)) = \psi^{-1}(1 - 1) = \psi^{-1}(0) = 0$. To show that N is strict, let $x < y$. Then, $\psi(x) < \psi(y)$
$\Longrightarrow 1 - \psi(y) < 1 - \psi(x)$
$\Longrightarrow \psi^{-1}(1 - \psi(y)) = N(y) < \psi^{-1}(1 - \psi(x)) = N(x)$.

Finally, to show that N is strong, we show that N is a self-involution. Take $N_1 = 1 - x$. Then, $N^{-1} = \left[\psi^{-1} N_1 \psi\right]^{-1} = \psi^{-1}(N_1)^{-1}\left(\psi^{-1}\right)^{-1} = \psi^{-1} N_1^{-1} \psi = \psi^{-1} N_1 \psi = N$ since $N_1^{-1}(x) = N(x) = 1 - x$ ∎

It follows that every strong negation N can be expressed as $N(x) = \psi^{-1}(1 - \psi(x))$, where ψ is an automorphism on [0.1], which is called a generator of N. The strong negation N with the generator ψ will be denoted by N_ψ. Generally speaking, generator of a strong negation is not unique. For example, both $\psi(x) = x$ and

$$\psi(x) = \begin{cases} \sqrt{x/2} & x < 0.5 \\ 1 - \sqrt{\frac{1-x}{2}} & x \geq 0.5 \end{cases}$$

are generators of the standard negation $N(x) = 1 - x$

Definition 195 *A mapping* $T : [0,1] \times [0,1] \longrightarrow [0,1]$ *is called a* ***triangular norm*** *(t-norm) or a conjunction, if it satisfies:*
(1) symmetry:$T(x,y) = T(y,x)$ *whenever* $x, y \in [0,1]$*;*
(2) monotonicity:$T(x_1,y_1) \leq T(x_2,y_2)$ *whenever* $x_1 \leq x_2$ *and* $y_1 \leq y_2$*;*
(3) associativity:$T(T(x,y),z) = T(x,T(y,z))$ *whenever* $x, y, z \in [0,1]$*;*
(4) boundary condition:$T(1,x) = x$ *whenever* $x \in [0,1]$*.*

Example 196 $T_{\min}(x,y) = x \wedge y$

Example 197 $T_L(x,y) = \max\{0, x + y - 1\}$ *(Lukasiewicz t-norm)*

Example 198 $T_0(x,y) = \begin{cases} x & \textit{if } y = 1 \\ y & \textit{if } x = 1 \\ 0 & \textit{otherwise} \end{cases}$

Example 199 $T_\pi(x,y) = xy$

Definition 200 *Let* φ *be an automorphism on* $[0,1]$ *and* T *a t-norm. Define* $T^\varphi(x,y) = \varphi^{-1} T(\varphi(x), \varphi(y))$ $\forall x, y \in [0,1]$*. Then,* T^φ *is a t-norm, called* $\varphi-$***transform*** *of* T

Proposition 201 *$T^{\varphi}(x,y)$ is a t-norm*

Proof. (1) $T^{\varphi}(x,y) = \varphi^{-1}T(\varphi(x), \varphi(y)) = \varphi^{-1}T(\varphi(y), \varphi(x)) = T^{\varphi}(y,x)$

(2) $x_1 \leq x_2$ and $y_1 \leq y_2$, then $\varphi(x_1) \leq \varphi(x_2)$ and $\varphi(y_1) \leq \varphi(y_2)$ so that $T(\varphi(x_1), \varphi(y_1)) \leq T(\varphi(x_2), \varphi(y_2))$

$\implies \varphi^{-1}T(\varphi(x_1), \varphi(y_1)) \leq \varphi^{-1}T(\varphi(x_2), \varphi(y_2))$

$\implies T^{\varphi}(x_1, y_1) \leq T^{\varphi}(x_2, y_2)$

(3) $T^{\varphi}(T^{\varphi}(x,y), z)$

$= \varphi^{-1}T\left(\varphi\varphi^{-1}T(\varphi(x), \varphi(y)), \varphi(z)\right)$

$= \varphi^{-1}T(\varphi(x), (T\varphi(y), \varphi(z)))$

$= \varphi^{-1}T\left(\varphi(x), \varphi\varphi^{-1}(T\varphi(y), \varphi(z))\right)$

$= \varphi^{-1}T(\varphi(x), \varphi(T^{\varphi}(x,y)))$

$= T^{\varphi}(x, T^{\varphi}(y,z))$

(4) $T^{\varphi}(1,x)$

$= \varphi^{-1}(T(\varphi(1), \varphi(x)))$

$= \varphi^{-1}(T(1, \varphi(x)))$

$= \varphi^{-1}\varphi(x) = x$ ■

Proposition 202 $T_0 \leq T_L \leq T_\pi \leq T_{\min}$

Proof. If $x = y = 1$, then the inequality holds. If $y = 1$ for any x and $x = 1$ for any y, then the inequality holds. Assume that $x, y \neq 1$. Then, $T_0(x,y) = 0 \leq \max\{0, x+y-1\} \leq xy \leq \min\{x,y\} = T_{\min}(x,y)$ ■

Proposition 203 *$T_0 \leq T \leq T_{\min}$ holds for any t-norm T.*

Proof. $\forall x, y \in [0,1]$, $x \leq x$ and $y \leq 1$ so that $T(x,y) \leq T(x,1) = x$. Similarly, $x \leq 1, y \leq y$ so that $T(x,y) \leq T(1,y) = y$. Hence, $T(x,y) \leq x \wedge y = T_{\min}(x,y)$. If $x = 1$ or $y = 1$, then $T(x,y) = T_0(x,y)$ by boundary condition and symmetry. If $x, y \neq 1$, then $T_0(x,y) = 0 \leq T(x,y)$. In any case $T_0 \leq T$ ■

So the set of all t-norms is bounded with the greatest t-norm $T_{\min}$ and the least t-norm T_0.

Proposition 204 $T(x,0) = 0\ \forall x$

Proof. $T(x,y) \geq 0$ holds trivially. For converse, note that since for any T, we have $T_0 \leq T \leq T_{\min}$ and $T_{\min}(x,0) = 0$, therefore $T(x,0) \leq 0$ ■

Proposition 205 *If a t-norm satisfies idempotency: $T(x,x) = x\ \forall x \in [0,1]$, then $T = T_{\min}$.*

Proof. For any T, we must have $T \leq T_{\min}$. To prove the converse, $T_{\min}(x,y) = x \wedge y = T(x \wedge y, x \wedge y) \leq T(x,y)$ since $x \geq x \wedge y$ and $y \geq x \wedge y$. ■

Definition 206 *A mapping S from $[0,1] \times [0,1]$ to $[0,1]$ is called a **triangular conorm** (t-conorm) or a disjunction, if it satisfies:*

(1) symmetry: $S(x,y) = S(y,x)$ whenever $x, y \in [0,1]$;

(2) monotonicity: $S(x_1,y_1) \leq S(x_2,y_2)$ whenever $x_1 \leq x_2$ and $y_1 \leq y_2$;

(3) associativity: $S(S(x,y),z) = S(x, S(y,z))$ whenever $x,y,z \in [0,1]$;

(4) boundary condition: $S(0,x) = x$ whenever $x \in [0,1]$.

Remark 207 *Let T be a t-norm and S be a t-conorm. From an algebraic view, both $([0,1],T)$ and $([0,1],S)$ are semigroups with identities 1 and 0 respectively, and thus they are commutative monoids.*

Definition 208 *Let ϕ be an automorphism on $[0,1]$ and S a t-conorm. Define S^{ϕ} by: $\forall x,y\in[0,1]$ such that $S^{\phi}(x,y)=\phi^{-1}S(\phi(x),\phi(y))$. This is the $\phi-$**transform** of the t-conorm*

Proof. (1) $S^{\varphi}(x,y)=\varphi^{-1}S(\varphi(x),\varphi(y))=\varphi^{-1}S(\varphi(y),\varphi(x))=S^{\varphi}(y,x)$

(2) $x_1\le x_2$ and $y_1\le y_2$, then $\varphi(x_1)\le\varphi(x_2)$ and $\varphi(y_1)\le\varphi(y_2)$ so that $S(\varphi(x_1),\varphi(y_1))\le S(\varphi(x_2),\varphi(y_2))$

$\Longrightarrow\ \varphi^{-1}S(\varphi(x_1),\varphi(y_1))\le\varphi^{-1}S(\varphi(x_2),\varphi(y_2))$

$\Longrightarrow\ S^{\varphi}(x_1,y_1)\le S^{\varphi}(x_2,y_2)$

(3) $S^{\varphi}(S^{\varphi}(x,y),z)$

$=\varphi^{-1}S\left(\varphi\varphi^{-1}S(\varphi(x),\varphi(y)),\varphi(z)\right)$

$=\varphi^{-1}S(\varphi(x),(S\varphi(y),\varphi(z)))$

$=\varphi^{-1}S\left(\varphi(x),\varphi\varphi^{-1}(S\varphi(y),\varphi(z))\right)$

$=\varphi^{-1}S(\varphi(x),\varphi(S^{\varphi}(x,y)))$

$=S^{\varphi}(x,S^{\varphi}(y,z))$

(4) $S^{\varphi}(0,x)$

$=\varphi^{-1}(S(\varphi(0),\varphi(x)))$

$=\varphi^{-1}(S(0,\varphi(x)))$

$=\varphi^{-1}\varphi(x)=x$ ■

Proposition 209 *Let T be a t-norm and η a strict negation. Define $S:[0,1]\times[0,1]\longrightarrow[0,1]$ $S(x,y)=\eta^{-1}T(\eta(x),\eta(y))$. Then S is a t-conorm*

Proof. (1) $S(x,y)=\eta^{-1}T(\eta(x),\eta(y))=\eta^{-1}T(\eta(y),\eta(x))=S(y,x)$

(2) $x_1\le x_2$ and $y_1\le y_2$, then $\eta(x_2)\le\eta(x_1)$ and $\eta(y_2)\le\eta(y_1)$ so that $T(\eta(x_2),\eta(y_2))\le T(\eta(x_1),\eta(y_1))$

$\Longrightarrow\ \eta^{-1}T(\eta(x_1),\eta(y_1))\le\eta^{-1}T(\varphi(x_2),\varphi(y_2))$

$\Longrightarrow\ S(x_1,y_1)\le S(x_2,y_2)$

(3) $S(S(x,y),z)$

$=\eta^{-1}T\left(\eta\eta^{-1}T(\eta(x),\eta(y)),\eta(z)\right)$

$=\eta^{-1}T(\eta(x),(T\eta(y),\eta(z)))$

$=\eta^{-1}T\left(\eta(x),\eta\eta^{-1}(T\eta(y),\eta(z))\right)$

$=\eta^{-1}T(\eta(x),\eta(S(x,y)))$

$=S(x,S(y,z))$

(4) $S(0,x)$

$=\eta^{-1}(T(\eta(0),\eta(x)))$

$=\eta^{-1}(T(1,\eta(x)))$

$=\eta^{-1}(\eta(x))=x$ ■

Thus, we can construct the following t-conorms from the given t-norm

Example 210 $S_{\max}(x,y)=\eta^{-1}T_{\min}(\eta(x),\eta(y))$

$=\eta^{-1}(\eta(x)\wedge\eta(y))$

$=\eta^{-1}((1-x)\wedge(1-y))$

$= \eta^{-1}(\min\{1-x, 1-y\})$
$= 1 - \min\{1-x, 1-y\}$
$= \max\{x, y\}$

Example 211 $S_L(x,y) = \eta^{-1} T_L(\eta(x), \eta(y))$
$= \eta^{-1}(\max\{0, \eta(x) + \eta(y) - 1\})$
$= \eta^{-1}(\max\{0, 1-x+1-y-1\})$
$= \eta^{-1}(\max\{0, 1-x-y\})$
$= 1 - \max\{0, 1-x-y\}$
$= \min\{1, x+y\}$

Example 212 $S_0(x,y) = \eta^{-1} T_0(\eta(x), \eta(y)) = \eta^{-1}\left(\left\{\begin{array}{cc} \eta(x) & \textit{if } \eta(y) = 1 \\ \eta(y) & \textit{if } \eta(x) = 1 \\ 0 & \textit{otherwise} \end{array}\right.\right)$
$= \eta^{-1}\left(\left\{\begin{array}{cc} \eta(x) & \textit{if } y = 0 \\ \eta(y) & \textit{if } x = 0 \\ 0 & \textit{otherwise} \end{array}\right.\right) = \left\{\begin{array}{cc} x & \textit{if } y = 0 \\ y & \textit{if } x = 0 \\ 1 & \textit{otherwise} \end{array}\right.$

Example 213 $S_\pi(x,y) = \eta^{-1} T_\pi(\eta(x), \eta(y)) = \eta^{-1}(\eta(x)\eta(y))$
$= \eta^{-1}[(1-x)(1-y)] = \eta^{-1}[1-(x+y-xy)] = x+y-xy$

Proposition 214 $S_0 \geq S_L \geq S_\pi \geq S_{\max}$

Proof. Since $T_0(\eta(x), \eta(y)) \leq T_L(\eta(x), \eta(y)) \leq T_\pi(\eta(x), \eta(y)) \leq T_{\min}(\eta(x), \eta(y))$ and η^{-1} reverses order so we can apply this to complete the proof. ■

Proposition 215 $S_0 \geq S \geq S_{\max}$

Proof. Again, since $T_0(\eta(x), \eta(y)) \leq T(\eta(x), \eta(y)) \leq T_{\min}(\eta(x), \eta(y))$ for any T, therefore $S_0 \geq S \geq S_{\max}$ ■

Proposition 216 $S(x,1) = 1$

Proof. $S(x,1) = \eta^{-1}(T(\eta(x), \eta(1))) = \eta^{-1}(T(\eta(x), 0)) = \eta^{-1}(0) = 1$ ■

Proposition 217 *If a t-conorm S satisfies idempotency, then $S = S_{\max}$*

Proof. If $S(x,x) = x = \eta^{-1} T(\eta(x), \eta(x))$
Then, $x = \eta^{-1} T(\eta(x), \eta(x))$
$\implies \eta(x) = T(\eta(x), \eta(x))$
$\implies x = T(x,x)$
$\implies T = T_{\min}$
$\implies S = S_{\max}$ ■
We have the following propositions concerning the absorption law:

Proposition 218 *Let T and S be a t-norm and a t-conorm respectively. If $\forall x, y \in [0,1]$, $T(S(x,y), x) = x$, then $T = T_{\min}$.*

Proof. Since $T(S(x,y),x) = x \; \forall x, y$, it is particularly valid for $y = 0$ in which case we have the idempotent law $T(S(x,0),x) = T(x,x) = x$ so that $T = T_{\min}$ ■

Proposition 219 *Let T and S be a t-norm and a t-conorm respectively. If $\forall x, y \in [0,1]$, $S(T(x,y),x) = x$, then $S = S_{\max}$*

Proof. Again, we choose $y = 1$ to get $S(T(x,1),x) = S(x,x) = x$ so that $S = S_{\max}$ ■

Another similar proposition holds for the distributive law

Proposition 220 *Let T and S be a t-norm and a t-conorm respectively. If $\forall x, y, z \in [0,1]$, $S(x, T(y,z)) = T(S(x,y), S(x,z))$, then $T = T_{\min}$.*

Proof. If we let $z = 0$, then $S(x, T(y,0)) = T(S(x,y), S(x,0))$
$\implies S(x,0) = T(S(x,y),x)$
$\implies x = T(S(x,y),x)$
$\implies T = T_{\min}$ ■

Proposition 221 *Let T and S be a t-norm and a t-conorm respectively. If $\forall x, y, z \in [0,1]$, $T(x, S(y,z)) = S(T(x,y), T(x,z))$, then $S = S_{\max}$.*

Proof. Take $z = 1$. Then, $T(x, S(y,1)) = S(T(x,y), T(x,1))$
$\implies T(x,1) = S(T(x,y),x)$
$x = S(T(x,y),x)$
$\implies S = S_{\max}$ ■

Thus, the distributive laws imply the absoprtion laws which in turn imply the idempotent law.

Definition 222 *Let T and S be a t-norm and a t-conorm respectively and η a strict negation. If $\forall x \in [0,1]$, $\eta(S(x,y)) = T(\eta(x), \eta(y))$, then (T, S, η) is called a* ***De Morgan triple.***

Definition 223 *Let A, B be fuzzy sets on X and (T, S, η) a De Morgan triple. The* ***complement*** *A_η^c of A under η, the intersection $A \cap_T B$ of A and B under t-norm T and union $A \cup_S B$ of A and B under t-conorm S are respectively defined by: $\forall x \in X$, $A_\eta^c(x) = \eta A(x)$, $(A \cap_T B)(x) = T(A(x), B(x))$ and $(A \cup_S B)(x) = S(A(x), B(x))$.*

If $T = T_{\min}$, $\eta(x) = 1 - x$ and $S = S_{\max}$, then $(A \cap_T B)(x) = A(x) \wedge B(x)$ and $(A \cup_S B)(x) = A(x) \vee B(x)$, which are Zadeh's intersection and union. If $T = T_\pi$, η and $S = S\pi$, then $(A \cap_T B)(x) = A(x)B(x)$ and $(A \cup_S B)(x) = A(x) + B(x) - A(x)B(x)$. If $T = T_L$, $\eta(x) = 1 - x$ and $S = S_L$, then $(A \cap TB)(x) = \max\{0, A(x) + B(x) - 1\}$ and $(A \cup_S B)(x) = \min\{1, A(x) + B(x)\}$.

Proposition 224 *If (T, S, η) is a De Morgan triple, then the algebraic system $(F(X), \cup_S, \cap_T, c)$ has the following properties:*
(1) $A \cap_T B \subseteq A \subseteq A \cup_S B$;

(2) $A\cap_T B = B\cap_T A, A\cup_S B = B\cup_S A$
(3) $(A\cap_T B)\cap_T C = A\cap_T (B\cap_T C), (A\cup_S B)\cup_S C = A\cup_S (B\cup_S C)$;
(4) $A\cap_T \varnothing = \varnothing, A\cup_S \varnothing = A, A\cap_T X = A, A\cup_S X = X$;
(5) $(A\cup_S B)^c_\eta = A^c_\eta \cap_T B^c_\eta$. *If* η *is a strong negation, then* $(A\cap_T B)^c_\eta = A^c_\eta \cup_S B^c_\eta$

Proof. This proof is valid $\forall x$ so that the propositions hold.
(1) $(A\cap_T B)(x) = T(A(x), B(x))$
$\leq T_{\min}(A(x), B(x)) \leq A(x) = S(A(x), 0) \leq S(A(x), B(x))$
(2) $(A\cap_T B)(x) = T(A(x), B(x)) = T(B(x), A(x)) = B\cap_T A$
$(A\cup_S B)(x) = S(A(x), B(x)) = S(B(x), A(x)) = (B\cup_S A)(x)$
(3) $(A\cap_T B)\cap_T C = T(T(A(x), B(x)), C(x)) = T(A(x), T(B(x), C(x))) = A\cap_T (B\cap_T C)$
$(A\cup_S B)\cup_S C = S(S(A(x), B(x)), C(x)) = S(A(x), S(B(x), C(x))) = A\cup_S (B\cup_S C)$;
(4) $(A\cap_T \varnothing)(x) = T(A(x), \varnothing(x)) = T(A(x), 0) = 0 = \varnothing(x)$
$(A\cup_S \varnothing)(x) = S(A(x), \varnothing(x)) = S(A(x), 0) = A(x)$
$(A\cap_T X)(x) = T(A(x), X(x)) = T(A(x), 1) = A(x)$
$(A\cup_S X)(x) = S(A(x), X(x)) = S(A(x), 1) = 1 = X(x)$
(5) $(A\cup_S B)^c_\eta(x) = \eta((A\cup_S B)(x)) = T(\eta(A(x)), \eta(B(x))) = T(A^c_\eta(x), B^c_\eta(x)) = (A^c_\eta \cap_T B^c_\eta)(x)$.
If η is a strong negation, $(A\cap_T B)^c\eta(x) = \eta((A\cap_T B)(x)) = \eta(T(A(x), B(x))) = \eta(\eta(S(A^c_\eta(x), B^c_\eta(x)))) = (A^c_\eta \cup_S B^c_\eta)(x)$. ■

Definition 225 *Let* $I : [0,1]\times[0,1] \longrightarrow [0,1]$. *If* $I(x,y)$ *is decreasing in* x *and increasing in* y *(usually* I *is called hybrid monotonous) and satisfies* $I(1,0) = 0, I(0,0) = I(1,1) = 1$*, then* I *is called a* ***fuzzy implication.***

A fuzzy implication is an extension of the ordinary implication in classic logic.

Proposition 226 $I(0,1) = 1$

Proof. For $0 \leq 1$, $I(1,1) \leq I(0,1)$
That is, $1 \leq I(0,1) \implies I(0,1) = 1$ ■

Example 227 $I_1(x,y) = \max(1-x, y)$

Example 228 $I_2(x,y) = \min(1-x+y, 1)$

Example 229 $I_3(x,y) = \begin{cases} 1 & x \leq y \\ y/x & x > y \end{cases}$ 1

Proposition 230 *If* I *is a fuzzy implication and* η *is a negation, then* I *defined by* $\forall x, y \in [0,1]$, $\hat{I}(x,y) = I(\eta(y), \eta(x))$ *is a fuzzy implication as well.*

Proof. For $x_1 \leq x_2$, $\eta(x_2) \leq \eta(x_1)$ and $y_1 \leq y_2$, $\eta(y_2) \leq \eta(y_1)$ so that $\hat{I}(x_2, y_1) = I(\eta(y_1), \eta(x_2)) \leq I(\eta(y_2), \eta(x_1)) = \hat{I}(x_1, y_2)$

$\hat{I}(1,0) = I(\eta(0), \eta(1)) = I(1,0) = 0$

$\hat{I}(1,1) = I(\eta(1), \eta(1)) = I(0,0) = 1$

$\hat{I}(0,0) = I(\eta(0), \eta(0)) = I(1,1) = 1$ ■

Proposition 231 *A mapping* $I : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a fuzzy implication iff it satisfies the following:*

$(I_1) \forall x \leq z, I(x,y) \geq I(z,y);$

$(I_2) \forall y \leq z, I(x,y) \leq I(x,z);$

$(I_3) \forall x \in [0,1], I(0,x) = 1;$

$(I_4) \forall x \in [0,1], I(x,1) = 1;$

$(I_5) I(1,0) = 0.$

Definition 232 *Let* S *be a* t*-conorm and* N *a strong negation. Then,* I *defined by* $\forall x \in [0,1]$*,* $I(x,y) = S(N(x),y)$ *is called an* S***-implication.***

Proposition 233 $I(x,y) = S(N(x),y)$ *is a fuzzy implication.*

Example 234 $S = S_{\max}, N(x) = 1 - x, I(x,y) = \max(1-x, y).$

Example 235 $S = S_{\pi}, N(x) = 1 - x, I(x,y) = 1 - x + xy$

Example 236 $S = S_L, N(x) = 1 - x, I(x,y) = \min(1 - x + y, 1).$

Theorem 237 *An implication* I *is an* S*-implication iff* I *satisfies the following properties:*

(1) $\forall x \in [0,1]$*,* $I(1,x) = x$ *(the so-called neutrality principle);*

(2) $\forall x, y, z \in [0,1]$ $I(x, I(y,z)) = I(y, I(x,z))$ *(the so-called exchange principle);*

(3) There exists a strong negation N *such that* $\forall x, y \in [0,1], I(x,y) = I(N(y), N(x))$*.*

Proof. Necessity. If $I(x,y) = S(N(x),y))$,where S is t-conorm and N a strong negation, then $I(1,x) = S(0,x) = x$, and thus (1) is valid. In addition, $I(x, I(y,z)) = S(N(x), I(y,z)) = S(N(x), S(N(y),z)) = S(S(N(y),z), N(x)) = S(N(y), S(z, N(x))) = I(y, I(x,z))$. Hence (2) is true. Finally, $I(N(y), N(x)) = S(y, N(x)) = I(x,y)$, i.e. (3) is true.

Sufficiency. Suppose that I satisfies (1), (2) and (3). Let $S(x,y) = I(N(x),y)$. We prove that S is a t-conorm. Firstly, $S(0,y) = I(N(0), y) = I(1,y) = y$, i.e. the boundary condition is satisfied. Since $S(x,y) = I(N(x),y) = I(N(y), N(N(x))) = I(N(y),x) = S(y,x)$ by (3), S is symmetric. Meanwhile, $\forall x, y, z \in [0,1], S(x, S(y,z)) = I(N(x), S(y,z)) = I(N(x), I(N(y),z)) = I(N(x), I(N(z),y))$ (by (3)) $= I(N(z), I(N(x),y))$ (by (2)) $= I(N(I(N(x),y)),z)$ (by (3)) $= I(N(S(x,y)),z) = S(S(x,y),z)$. Hence S is associative. In summary, S is a t-conorm. Noticing that $I(x,y) = S(N(x),y)$, I is an S-implication. ■

Definition 238 *Let T be a t-norm. Then I_T defined by: $\forall x, y \in [0,1]$, $I_T(x,y) = \sup\{z \mid T(x,z) \leq y\}$ is called an **R-implication.***

The definition is based on the following equality of crisp sets: $A^c \cup B = (A \backslash B)^c = \cup\{X | A \cap X \subseteq B\}$. It should be pointed out that I_T is indeed a fuzzy implication.

Proof. $I_T(0,0) = \sup\{z \mid T(0,z) = 0 \leq y\} = 1$

$I_T(1,1) = \sup\{z \mid T(1,z) = z \leq y\} = 1$

$I_T(1,0) = \sup\{z \mid T(1,z) = z \leq 0\} = 0$

To show that I is decreasing in x and increasing in y, take $x_2 \leq x_1$ and $y_1 \leq y_2$

Then, $I_T\,(x_1, y_1)$

$= \sup\{z \mid T(x_1, z) \leq y_1\}$

$= \sup\{z \mid T(x_2, z) \leq T(x_1, z) \leq y_1\}$

$= \sup\{z \mid T(x_2, z) \leq y_1\}$

$\leq \sup\{z \mid T(x_2, z) \leq y_2\}$

$= I_T\,(x_2, y_2)$ ■

Example 239 *Let $T = T_{\min}$. Then we have the Godel implication:* $I_T(x,y) = \begin{cases} 1 & x \leq y \\ y & x > y \end{cases}$

Example 240 *Let $T = T_\pi$. Then we have the Goguen implication:* $I_T(x,y) = \begin{cases} 1 & x \leq y \\ y/x & x > y \end{cases}$

Example 241 *Let $T = T_L$. Then we have the Lukasiewicz implication: $I_T(x,y) = \min(1 - x + y, 1)$.*

Definition 242 *A mapping $E : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a **fuzzy equivalence** if it satisfies that*

(E1)$\forall x, y \in [0,1], E(x,y) = E(y,x)$;

(E2) $E(0,1) = E(1,0) = 0$;

(E3) $\forall x \in [0,1], E(x,x) = 1$;

(E4) $E(x,y) \leq E(x_1, y_1)$ whenever $x \leq x_1 \leq y_1 \leq y$.

Example 243 *Godel equivalence:* $E(x,y) = \begin{cases} 1 & x = y \\ \min(x,y) & x \neq y \end{cases}$

Example 244 *Goguen equivalence:* $E(x,y) = \begin{cases} 1 & x = y = 0 \\ \frac{\min(x,y)}{\max(x,y)} & \text{otherwise} \end{cases}$

Example 245 *Lukasiewicz equivalence: $E(x,y) = 1 - |x - y|$*

Proposition 246 *The mapping $E : [0,1] \times [0,1] \longrightarrow [0,1]$ is a fuzzy equivalence iff there exists a fuzzy implication I such that $\forall x \in [0,1], I(x,x) = 1$ and $E(x,y) = \min(I(x,y), I(y,x))$.*

Proof. ($\Longrightarrow$) Let $I(x,y) = \begin{cases} 1 & x \leq y \\ E(x,y) & x > y \end{cases}$. We verify that I is a fuzzy implication. Firstly, we show that $I(x,y) \geq I(z,y)$ whenever $x \leq z$. If $x \leq y, I(x,y) = 1$ and the desired equality trivially holds. If $x > y$, then $y < x \leq z$, $I(x,y) = E(x,y) \geq E(z,y) = I(z,y)$. Hence $I(\cdot, y)$ is decreasing in x. To show that $I(x,\cdot)$ is increasing in y, assume $y \leq z$. If $y \geq x$, then $x \leq y$ and $x \leq z$ so that $I(x,y) = 1 \leq 1 = I(x,z)$. If $x > y$, then either $z \geq x > y$ or $x > z \geq y$. In the first case, $I(x,z) = 1 \geq I(x,y)$. In the second case, $I(x,z) = E(x,z) \geq E(x,y) = I(x,y)$.

$I(0,1) = 1$

$I(1,1) = 1$

$I(0,0) = 1$ trivially hence I is an implication.

When $x \leq y$, $I(x,y) = 1$, and $I(y,x) = E(y,x) = E(x,y) = I(x,y)$. Hence, $E(x,y) = \min(I(x,y), I(y,x))$. In the case of $x > y$, then $I(x,y) = E(x,y) = E(y,x) = I(y,x)$ and hence $E(x,y) = \min(I(x,y), I(y,x))$

($\Longleftarrow$) $E(x,y) = \min(I(x,y), I(y,x)) = \min(I(y,x), I(x,y)) = E(y,x)$.

$E(x,x) = \min(I(x,x), I(x,x)) = \min(I(x,x)) = I(x,x) = 1$

$E(0,1) = \min(I(0,1), I(1,0)) = \min(1,0) = 0 = E(1,0)$ by symmetry of E

Take $x \leq x_1 \leq y_1 \leq y$. Then, $E(x,y) = \min(I(x,y), I(y,x))$

$\leq \min(I(x,y), I(y_1,x_1))$

$\leq I(y_1,x_1)$

Notice that $I(y_1,x_1) \leq I(x_1,y_1)$ so that $I(y_1,x_1) \leq \min(I(x_1,y_1), I(y_1,x_1)) = E(x_1,y_1)$ ■

Corollary 247 *A mapping $E : [0,1] \times [0,1] \longrightarrow [0,1]$ is a fuzzy equivalence iff there exists a fuzzy implication I such that $\forall x \in [0,1]$, $I(x,x) = 1$ and $E(x,y) = I(\max(x,y), \min(x,y))$*

Proof. ($\Longleftarrow$) $E(x,x) = I(\max(x,x), \min(x,x)) = I(x,x) = 1$

$E(0,1) = I(\max(0,1), \min(0,1)) = I(1,0) = 0$

$E(1,0) = I(\max(1,0), \min(1,0)) = I(1,0) = 0$

Let $x \leq x_1 \leq y_1 \leq y$. Then,

$E(x,y) = I(\max(x,y), \min(x,y)) \leq I(\max(x_1,y_1), \min(x_1,y_1)) = E(x_1,y_1)$

($\Longrightarrow$) Consider $E(x,y) = I(\max(x,y), \min(x,y)) = \min(I(x,y), I(y,x))$.

Then, $E(x,x) = 1 = I(x,x)$

Clearly, $I(0,0) = I(1,1) = 1$

Next, $E(0,1) = 0 = I(1,0)$ and also $E(1,0) = I(0,1) = 0$

Finally, let $x \leq x_1 \leq y_1 \leq y$

Then, $E(x,y) \leq E(x_1,y_1)$

$\Longrightarrow$ $I(\max(x,y), \min(x,y)) = I(y,x) \leq I(\max(x_1,y_1), \min(x_1,y_1)) = I(y_1,x_1)$ so that I is decreasing in x and increasing in y. ■

Definition 248 *Let A be a fuzzy set on X. For $\alpha \in [0,1]$, the α-**cut** A_α of A is defined as $A_\alpha = \{x | A(x) \geq \alpha\}$, and the **strong** α-**cut** $\hat{A}_\alpha$ of A is defined as $A_\alpha. = \{x | A(x) > \alpha\}$*

Proposition 249 $\hat{A}_\alpha \subseteq A_\alpha\ (\forall \alpha \in [0,1]),\ A_0 = X,\ \hat{A}_1 = \varnothing,\ A_1 = \ker(A)$ *and* $\hat{A}_0 = supp(A)$

Proof. If $x \in \hat{A}_\alpha$, then $x > \alpha \implies x \geq \alpha \implies x \in A_\alpha$

$A_0 = \{x | A(x) \geq 0\} = X$

$A_1 = \{x | A(x) > 1\} = \varnothing$

$A_1 = \{x | A(x) \geq 1\} = \{x | A(x) = 1\} = \ker(A)$

$A_0 = \{x | A(x) > 0\} = supp(A)$ ∎

Proposition 250 *Let* $A, B, A_i, B_i \in F(X)\ (i \in I)$. *Then* $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$, $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$

Proof. If $x \in (A \cup B)_\alpha$, then $(A \cup B)(x) \geq \alpha$

$\implies A(x) \vee B(x) \geq \alpha$

$\implies A(x) \geq \alpha$ or $B(x) \geq \alpha$

$\implies x \in A_\alpha$ or $x \in B_\alpha$

Conversely, $x \in A_\alpha \cup B_\alpha$

$\implies x \in A_\alpha$ or $x \in B_\alpha$

$\implies A(x) \geq \alpha$ or $B(x) \geq \alpha$

$\implies A(x) \vee B(x) \geq \alpha$

$\implies (A \cup B)(x) \geq \alpha$

$\implies x \in (A \cup B)_\alpha$

Next, $x \in (A \cap B)_\alpha \iff (A \cap B)(x) \geq \alpha$

$\iff A(x) \geq \alpha$ and $B(x) \geq \alpha$

$\iff x \in A_\alpha \cap B_\alpha$ ∎

Proposition 251 *Let* $A, B, A_i, B_i \in F(X)\ (i \in I)$. $(\hat{A} \cup B)_\alpha = \hat{A}_\alpha \cup B_\alpha$, $(\hat{A} \cap B)_\alpha = \hat{A}_\alpha \cap B_\alpha$.

Proof. The proof is nearly same as above: If $x \in (\hat{A} \cup \hat{B})_\alpha$, then $(\hat{A} \cup \hat{B})(x) > \alpha$

$\implies \hat{A}(x) \vee \hat{B}(x) > \alpha$

$\implies \hat{A}(x) > \alpha$ or $\hat{B}(x) > \alpha$

$\implies x \in \hat{A}_\alpha$ or $x \in \hat{B}_\alpha$

Conversely, $x \in \hat{A}_\alpha \cup \hat{B}_\alpha$

$\implies x \in \hat{A}_\alpha$ or $x \in \hat{B}_\alpha$

$\implies \hat{A}(x) > \alpha$ or $\hat{B}(x) > \alpha$

$\implies \hat{A}(x) \vee \hat{B}(x) > \alpha$

$\implies (\hat{A} \cup \hat{B})(x) > \alpha$

$\implies x \in (\hat{A} \cup \hat{B})_\alpha$

$x \in (\hat{A} \cap \hat{B})_\alpha \iff (\hat{A} \cap \hat{B})(x) > \alpha$

$\iff \hat{A}(x) > \alpha$ and $\hat{B}(x) > \alpha$

$\iff x \in \hat{A}_\alpha \cap \hat{B}_\alpha$ ∎

Proposition 252 *Let* $A, A_i \in F(X)\ (i \in I) \bigcup_{i \in I} (A_i)_\alpha \subseteq \left(\bigcup_{i \in I} A_i \right)_\alpha$, $\left(\bigcap_{i \in I} A_i \right)_\alpha = \bigcap_{i \in I} (A_i)_\alpha$,

Proof. $x \in \bigcup_{i\in I}(A_i)_\alpha$

$\Longrightarrow$ $\exists i \in I$, such that $x \in (A_i)_\alpha$

$\Longrightarrow$ $\exists i \in I$, $A_i(x) \geq \alpha$

$\Longrightarrow$ $\sup_{i\in I} A_i(x) \geq \alpha$

$\Longrightarrow$ $x \in \bigcup_{i\in I} A_i$

Next, for $\left(\bigcap_{i\in I} A_i\right)_\alpha = \bigcap_{i\in I}(A_i)_\alpha$

$x \in \left(\bigcap_{i\in I} A_i\right)_\alpha \iff \left(\bigcap_{i\in I} A_i\right)(x) \geq \alpha$

$\iff$ $A_i\,(x) \geq \alpha \ \forall i \in I$

$\iff$ $x \in (A_i)_\alpha \ \forall i \in I$

$\iff$ $x \in \bigcap_{i\in I}(A_i)_\alpha \ \forall i \in I$ ■

The converse of the first part of the proposition does not hold:

Let $A_n (n = 1, 2, ...)$ be fuzzy sets on a universal set X, defined by $\forall x \in X$, $A_n(x) = 0.5 - 1/n + 1$. Put $\alpha = 0.5$. Then $\forall x \in X$ and $n = 1, 2, ...$, $A_n(x) < \alpha$ and hence $(A_n)_\alpha = \varnothing$. As a consequence, $\bigcup_{n\in I}^{\infty}(A_n)_\alpha = \varnothing$

However, $\left(\bigcup_{n\in I}^{\infty} A_n\right)_\alpha = \sup_n A_n(x) = 0.5$ and hence $x \in \bigcup_{n\in I}^{\infty}(A_n)_\alpha$ for every x. Therefore, $\bigcup_{n\in I}^{\infty}(A_n)_\alpha = X$. Clearly, $\bigcup_{i\in I}(A_i)_\alpha \neq \left(\bigcup_{i\in I} A_i\right)_\alpha$

Proposition 253 $\left(\bigcup_{i\in I}\hat{A}_i\right)_\alpha = \bigcup_{i\in I}\left(\hat{A}_i\right)_\alpha$, $\left(\bigcap_{i\in I}\hat{A}_i\right)_\alpha \subseteq \bigcap_{i\in I}(\hat{A}_i)_\alpha$

Proof. $x \in \bigcup_{i\in I}(\hat{A}_i)_\alpha$

$\iff$ $\exists i \in I$, such that $x \in (\hat{A}_i)_\alpha$

$\iff$ $\exists i \in I$, $\hat{A}_i(x) > \alpha$

$\iff$ $\sup_{i\in I} A_i(x) > \alpha$

$\iff$ $x \in \bigcup_{i\in I}\hat{A}_i$

Next, for $\left(\bigcap_{i\in I}\hat{A}_i\right)_\alpha \subseteq \bigcap_{i\in I}(\hat{A}_i)_\alpha$

$x \in \left(\bigcap_{i\in I}\hat{A}_i\right)_\alpha \Longrightarrow \left(\bigcap_{i\in I}\hat{A}_i\right)(x) > \alpha$

$\implies \hat{A}_i(x) > \alpha \ \forall i \in I$
$\implies x \in \left(\hat{A}_i\right)_\alpha \ \forall i \in I$
$\implies x \in \bigcap_{i \in I} \left(\hat{A}_i\right)_\alpha \ \forall i \in I$ ■

The converse of the second part of the proposition does not hold:

Let $\hat{A}_n (n = 1, 2, ...)$ be fuzzy sets on a universal set X, defined by $\forall x \in X$, $\hat{A}_n(x) = 0.5 - 1/(n+1)$. Put $\alpha = 0.5$. Then $\forall x \in X$ and $n = 1, 2, ...$, $\hat{A}_n(x) < \alpha$ and hence $\left(\hat{A}_n\right)_\alpha = \varnothing$. As a consequence, $\bigcap_{i \in I}^{\infty} \left(\hat{A}_n\right)_\alpha = \varnothing$

However, $\left(\bigcap_{i \in I}^{\infty} A_n\right)_\alpha = \inf_n A_n(x) = 0.5$ and hence $x \in \bigcap_{i \in I}^{\infty} (A_n)_\alpha$ for every x. Therefore, $\bigcap_{i \in I}^{\infty} (A_n)_\alpha = X$. Clearly, $\bigcap_{i \in I}^{\infty} (A_i)_\alpha \neq \left(\bigcap_{i \in I}^{\infty} A_i\right)_\alpha$

Proposition 254 *Let $A, A_i \in F(X)$ $(i \in I)$. If $\alpha_1 < \alpha_2$, then $A_{\alpha_2} \subseteq A_{\alpha_1}$, $A_{\alpha_2} \subseteq \hat{A}_{\alpha_1}$ and $\hat{A}_{\alpha_2} \subseteq \hat{A}_{\alpha_1}$*

Proof. $x \in A_{\alpha_2}$
$\implies A(x) \geq \alpha_2 > \alpha_1$
$\implies A(x) \geq \alpha_1$ and $A(x) > \alpha_1$
$\implies x \in \hat{A}_{\alpha_1}$ and $x \in A_{\alpha_1}$
$x \in \hat{A}_{\alpha_2}$
$\implies A(x) > \alpha_2 > \alpha_1$
$\implies x \in \hat{A}_{\alpha_1}$ ■

Proposition 255 *Let $A, A_i \in F(X)$ $(i \in I)$. Let $\alpha = \bigvee_{i \in I} \alpha_i, \beta = \bigwedge_{i \in I} \alpha_i$. Then $\bigcap_{i \in I} A_{\alpha_i} = A_\alpha$, $\bigcup_{i \in I} A_{\alpha_i} = A_\beta$. Particularly, $\bigcap_{\lambda < \alpha} A_\lambda = A_\alpha$ and $\bigcup_{\lambda > \alpha} A_\lambda = A_\alpha$.*

Proof. $x \in \bigcap_{i \in I} A_{\alpha_i}$
$\iff A(x) \in A_{\alpha_i} \forall (i \in I)$
$\iff A(x) \geq \alpha_i \forall (i \in I)$
$\iff A(x) \geq \sup_i \alpha_i$
$\iff x \geq \alpha$
$\iff x \in A_\alpha$

For the second equality,
$x \in \bigcup_{i \in I} A_{\alpha_i}$
$\iff \exists i \in I$ such that $x \in A_{\alpha_i}$
$\iff \exists i \in I$ such that $x \geq \alpha_i$
$\iff A(x) \geq \inf_i \alpha_i$

$\iff A(x) \geq \beta$
$\iff x \in A_\beta$ ■

Proposition 256 *Let* $A, A_i \in F(X)$ $(i \in I)$. $(A^c)_\alpha = (A_{1-\alpha})^c$, $\left(\hat{A}^c\right)_\alpha = \left(\hat{A}_{1-\alpha}\right)^c$

Proof. $x \in (A_{1-\alpha})^c$
$\iff x \notin A_{1-\alpha}$
$\iff A(x) < 1-\alpha$
$\iff A^c(x) \geq \alpha$
$\iff x \in (A^c)_\alpha$
For the second,
$x \in \left(\hat{A}_{1-\alpha}\right)^c$
$\iff x \notin \hat{A}_{1-\alpha}$
$\iff \hat{A}(x) \leq 1-\alpha$
$\iff \hat{A}^c(x) > \alpha$
$\iff x \in \left(\hat{A}^c\right)_\alpha$ ■

Proposition 257 $\alpha_1 < \alpha_2$ *implies that* $\alpha_1 A \subseteq \alpha_2 A$

Proof. $\alpha_1 A(x) < \alpha_2 A(x)$ by hypothesis
so that $\alpha_1 A \subseteq \alpha_2 A$ ■

Proposition 258 $A_1 \subseteq A_2$ *implies that* $\alpha A_1 \subseteq \alpha A_2$

Proof. By hypothesis, $A_1(x) \leq A_2(x)$
so that $\alpha A_1(x) \leq \alpha A_2(x)$
$\Longrightarrow \alpha A_1 \subseteq \alpha A_2$ ■

Proposition 259 *For every* $A \in F(X)$, $A = \bigcup_{\alpha\in[0,1]} \alpha A_\alpha$

Proof. $\forall x \in X$,
$\left(\bigcup_{\alpha\in[0,1]} \alpha A_\alpha\right)(x) = \left(\bigvee_{\alpha\in[0,1]} \alpha A_\alpha\right)(x) = \left(\bigvee_{\alpha\in[0,1]} \alpha \wedge A_\alpha\right)$
$= \max\left\{\bigvee_{x\in A_\alpha} (\alpha \wedge A_\alpha(x)), \bigvee_{x\notin A_\alpha} (\alpha \wedge A_\alpha(x))\right\}$
$= \bigvee_{\alpha\leq A(x)} \alpha = A(x).$ ■

Corollary 260 *Let* A,$B \in F(X)$. *Then* $A = B \iff \forall\alpha \in [0,1],\ A_\alpha = B_\alpha$ $\iff \forall\alpha \in [0,1],\ \hat{A}_\alpha = \hat{B}_\alpha$

Proof. $A = B$

$\iff \bigcup_{\alpha\in[0,1]} \alpha A_\alpha = \bigcup_{\alpha\in[0,1]} \alpha B_\alpha$

$\iff \forall\alpha \in [0,1],\ A_\alpha = B_\alpha$

$\iff \forall\alpha \in [0,1],\ \hat{A}_\alpha = \hat{B}_\alpha$ ■

A fuzzy set and all its (strong) α-cuts thus are uniquely determined by each other. As a matter of fact, it can be seen that $A(x) = \bigvee_{x\notin A_\alpha} \alpha$ so we can find the fuzzy set A if its (strong) α-cuts are given for all $\alpha \in [0,1]$.

2.2 L-Fuzzy sets

In the definition of a fuzzy set, the range of the involved mapping is confined to the totally ordered set $[0,1]$. From the mathematical view, this restriction is not natural. In this section, $[0,1]$ is extended to a general lattice L, which leads to the so-called L-fuzzy sets. As in fuzzy sets, some operations such as union and intersection may be formed by employing the concept of supremum and infimum in L. However, a generalization of the complement operation needs some extra efforts since there is no operation in L available for formulating complement. In view of this, we start with the concept of a pseudo-complement.

Definition 261 *Let $(P,\leq)$ be a poset. A mapping $\eta : P \longrightarrow P$ such that*
(1) $\forall\alpha,\beta \in P$, $\alpha \leq \beta$ implies that $\eta(\beta) \leq \eta(\alpha)$,
(2) $\forall\alpha \in P$, $\eta(\eta(\alpha)) = \alpha$,
*is called a **pseudo-complement** on $(P,\leq)$*

Clearly, every strong negation is a pseudo-complement on $([0,1],\leq)$ and $\eta(A) = A^c$ $(\forall A \in \mathcal{P}(X))$ is a pseudo-complement on $(\mathcal{P}(X),\subseteq)$.

Example 262 *The complement c in a soft algebra $(L,\vee,\wedge,c)$ is a pseudocomplement. Since the complement c in every soft algebra is involutive, it suffices to prove that $\forall\alpha,\beta \in P$, $\alpha \leq \beta$ implies that $\beta^c \leq \alpha^c$. Indeed, when $\alpha \leq \beta$,*
$\beta^c \leq \beta^c \vee \alpha^c = (\beta \wedge \alpha)^c = \alpha^c$

The complement in a Boolean algebra is also a pseudo-complement since every Boolean algebra is a soft algebra.

Proposition 263 *If $(P,\leq)$ is a bounded poset with the greatest element 1 and the least element 0 and if η is a pseudo-complement on $(P,\leq)$, then $\eta(1) = 0$ and $\eta(0) = 1$.*

Proof. Trivially, $0 \leq \eta(1)$. Using this, we get $\eta(0) \geq \eta(\eta(1)) = 1$

Trivially, $\eta(0) \leq 1$. Using this, we get $0 = \eta(\eta(0)) \geq \eta(1)$.

Combining the four inequalities, by antisymmetry, we get $\eta(1) = 0$ and $\eta(0) = 1$. ■

Proposition 264 *If η is a pseudo-complement in a lattice $(L,\vee,\wedge)$, then $\eta(\alpha \vee \beta) = \eta(\alpha) \wedge \eta(\beta)$ and $\eta(\alpha \wedge \beta) = \eta(\alpha) \vee \eta(\beta)$*

Proof. It follows from $\alpha \geq \alpha \wedge \beta$ and $\beta \geq \alpha \wedge \beta$ that

$\eta(\alpha \wedge \beta) \geq \eta(\alpha) \vee \eta(\beta)$. Take $\alpha = \eta(x)$ and $\beta = \eta(y)$. Then, the last inequality gives us

$\eta(\eta(x) \wedge \eta(y)) \geq \eta(\eta(x)) \vee \eta(\eta(x)) = x \vee y$. Apply η on both sides again to get

$\eta\eta(\eta(x) \wedge \eta(y)) \leq \eta(x \vee y)$ so that $\eta(x) \wedge \eta(y) \leq \eta(x \vee y)$

In other words, $\eta(\alpha) \wedge \eta(\beta) \leq \eta(\alpha \vee \beta)$

By antisymmetry, $\eta(\alpha \vee \beta) = \eta(\alpha) \wedge \eta(\beta)$.

For the second, consider $\alpha \leq \alpha \vee \beta$ and $\beta \leq \alpha \vee \beta$ which gives us $\eta(\alpha) \geq \eta(\alpha \vee \beta)$ and

$\eta(\beta) \geq \eta(\alpha \vee \beta)$. Thus

$\eta(\alpha \vee \beta) \leq \eta(\alpha) \wedge \eta(\beta)$.

$\eta(\alpha \wedge \beta) \geq \eta(\alpha) \vee \eta(\beta)$. Next, we use $\eta(\alpha \vee \beta) \leq \eta(\alpha) \wedge \eta(\beta)$ and again take $\alpha = \eta(x)$ and

$\beta = \eta(y)$ to get $\eta(\eta(x) \vee \eta(y)) \leq \eta(\eta(x)) \wedge \eta(\eta(y)) = x \wedge y$

That is, $\eta(\eta(x) \vee \eta(y)) \leq x \wedge y$

Apply negation on both sides again to get $\eta\eta(\eta(x) \vee \eta(y)) \geq \eta(x \wedge y)$

or $\eta(x) \vee \eta(y) \geq \eta(x \wedge y)$

or $\eta(\alpha) \vee \eta(\beta) \geq \eta(\alpha \wedge \beta)$

Combining the two, we get the second equality ■

For a complete lattice, the preceding proposition can be extended as:

$\eta\left(\bigvee_{i \in I} \alpha_i\right) = \bigwedge_{i \in I} \eta(\alpha_i)$ and $\eta\left(\bigwedge_{i \in I} \alpha_i\right) = \bigvee_{i \in I} \eta(\alpha_i)$ where I is an arbitrary indexing set.

Definition 265 *Let X be the universe of discourse and let $(L, \vee, \wedge)$ be a lattice. A mapping $A : X \longrightarrow L$ is said to be an L-**fuzzy** set on X. The set of all L-fuzzy sets on X will be denoted by $F_L(X)$.*

$F_L(X)$ can be given whatever operations L has, and these operations will obey any law valid in L which extends point by point. For example, the concepts of subset, union and intersection can be defined by means of $\leq$, $\vee$ and $\wedge$ in L respectively. More specifically, let A,B$\in F_L(X)$. If $\forall x \in X$, $A(x) \leq B(x)$, then A is called a subset of B, denoted by $A \subseteq B$. The union $A \cup B$ of A and B is defined by $\forall x \in X$, $(A \cup B)(x) = A(x) \vee B(x)$. The intersection $A \cap B$ of A and B is defined by $\forall x \in X$, $(A \cap B)(x) = A(x) \wedge B(x)$. Clearly, $A = B$ iff $A \subseteq B$ and $B \subseteq A$. For a complete lattice $(L, \vee, \wedge)$ and $A_i \in F_L(X)$ $(i \in I)$, union and intersection can be extended, $\forall x \in X$, (

$\left(\bigcup_{i \in I} A_i\right)(x) = \bigvee_{i \in I} A_i(x)$ and $\left(\bigcap_{i \in I} A_i\right)(x) = \bigwedge_{i \in I} A_i(x)$

If there is a pseudo-complement η on $(L, \leq)$, then the complement A^c of A in $F_L(X)$ is defined by $\forall x \in X$, $A^c(x) = \eta(A(x))$.Generally speaking, $(F_L(X), \cup, \cap)$ is a lattice. As some additional conditions are imposed on L, $F_L(X)$ will gain some more properties. As examples, we have the following:

Proposition 266 *If $(L, \vee, \wedge)$ is a distributive lattice, then $(F_L(X), \cup, \cap)$ is a distributive lattice.*

Proof. First we prove that $(F_L(X), \cup, \cap)$ is a lattice. For $A, B, C \in F_L(X)$, and $\forall x \in X$, $(A \cup A)(x) = A(x) \vee A(x) = A(x)$. Thus, $A \cup A = A$. Similarly, $(A \cap A)(x) = A(x) \wedge A(x) = A(x)$

Next, $(A \cup B)(x) = A(x) \vee B(x) = B(x) \vee A(x) = (B \cup A)(x)$

Similarly, $(A \cap B)(x) = A(x) \wedge B(x) = B(x) \wedge A(x) = (B \cap A)(x)$

Furthermore, $(A \cup (B \cup C))(x) = A(x) \vee (B \cup C)(x) = A(x) \vee (B(x) \vee C(x))$
$= (A(x) \vee B(x)) \vee C(x)$
$= (A \cup B)(x) \vee C(x) = ((A \cup B) \cup C)(x)$

Similarly, $(A \cap (B \cap C))(x) = A(x) \wedge (B \cap C)(x) = A(x) \wedge (B(x) \wedge C(x))$
$= (A(x) \wedge B(x)) \wedge C(x)$
$= (A \cap B)(x) \wedge C(x) = ((A \cap B) \cap C)(x)$

Finally, $(A \cap (A \cup B))(x) = A(x) \wedge (A \cup B)(x) = A(x) \wedge (A(x) \vee B(x)) = A(x)$

And $(A \cup (A \cap B))(x) = A(x) \vee (A \cap B)(x) = A(x) \vee (A(x) \wedge B(x)) = A(x)$

To the prove the second part, since we have $A(x) \vee (B(x) \wedge C(x)) = (A(x) \vee B(x)) \wedge (A(x) \vee C(x))$

Therefore, $[A \cup (B \cap C)](x) = (A \cup B)(x) \cap (A \cup C)(x)$

Since the second distributive law is the equivalent as the first in a lattice, therefore the proof is complete. ■

Proposition 267 *Let $L = \mathcal{P}(X)$ and $\vee, \wedge$ and c be the union, intersection and complement of crisp sets respectively. Then $(F_L(X), \cup, \cap, c)$ is a Boolean algebra.*

2.3 Fuzzy Relations

As known to us, a relation is a subset of the Cartesian product of two sets. A relation is naturally fuzzified while a subset is fuzzified. In fact, whether two objects have a relation is not always easy to determine. For example, the relation "greater than" on the set of real numbers is a crisp one because we can determine the order relation of any two real numbers without vagueness. However, the relation "much greater than" is a fuzzy one because it is impossible for us to figure out the exact minimum difference of two numbers satisfying this relation. In real world problems, there exist a lot of such relations, e.g. " being friend of" and "being confident in" between some people. These relations will be termed as fuzzy relations.

Definition 268 *Let X and Y be two non-empty sets. A mapping $R : X \times Y \longrightarrow [0, 1]$ is called a* ***fuzzy (binary) relation*** *from X to Y. For $(x, y) \in X \times Y$, $R(x, y) \in [0, 1]$ is referred to as the degree of relationship between x and y. Particularly, a fuzzy relation from X to X is called a fuzzy (binary) relation on X.*

By definition, a fuzzy relation R is a fuzzy set on $X \times Y$, i.e. $R \in F(X \times Y)$. We know that the relation $>$ (greater than) on the set of real numbers is a crisp relation with the characteristic function defined by

$$> (x,y) = \begin{cases} 1 & x > y \\ 0 & \text{otherwise} \end{cases}$$

whereas the relation >> (much greater than) is a fuzzy relation on the set of real numbers, which may be expressed by

$$>> (x,y) = \begin{cases} 1 + \frac{100}{(x-y)^2} & x > y \\ 0 & \text{otherwise} \end{cases}$$

For instance, the ordered pairs $(x+1, x)$ have a low degree 1/101 with respect to ">>", the ordered pairs $(x+10, x)$ have an intermediate degree 0.5 with respect to ">>", and the ordered pairs $(x+100, x)$ have a high degree 100/101 with respect to ">>".

Definition 269 *Let R be a fuzzy relation from X to Y. The R-**afterset** xR of x ($x \in X$) is a fuzzy set on Y defined by $\forall y \in Y$, $(xR)(y) = R(x,y)$. The **R-foreset** Ry of y ($y \in Y$) is a fuzzy set on X defined by $\forall x \in X$, $(Ry)(x) = R(x,y)$.*

Since fuzzy relations are fuzzy sets, they have the same set-theoretic operations as fuzzy sets. Let R and S be fuzzy relations from X to Y. R is contained in S, denoted $R \subseteq S$, iff $\forall (x,y) \in X \times Y$, $R(x,y) \leq S(x,y)$; R is equal to S, denoted $R = S$, iff $\forall (x,y) \in X \times Y$, $R(x,y) = S(x,y)$. Clearly, $R = S$ iff $R \subseteq S$ and $S \subseteq R$. The union $R \cup S \in F(X \times Y)$ of R and S is defined by $\forall (x,y) \in X \times Y$, $(R \cup S)(x,y) = R(x,y) \vee S(x,y)$. The intersection $R \cap S \in F(X \times Y)$ of R and S is defined by $\forall (x,y) \in X \times Y$, $(R \cap S)(x,y) = R(x,y) \wedge S(x,y)$. The complement $R^c \in F(X \times Y)$ of R is defined by

$\forall (x,y) \in X \times Y$, $(R^c)(x,y) = 1 - R(x,y)$. The inverse $R^{-1} \in F(X \times Y)$ of R is defined by $\forall (x,y) \in X \times Y$, $R^{-1}(y,x) = R(x,y)$.

In addition, if $R_i \in F(X \times Y)$ for $i \in I$ indexing set, then $\bigcup_{i \in I} R_i$ is defined by $\forall (x,y) \in X \times Y$, $\left(\bigcup_{i \in I} R_i\right)(x,y) = \bigvee_{i \in I} R_i(x,y)$ and $\bigcap_{i \in I} R_i$ is defined by $\forall (x,y) \in X \times Y$, $\left(\bigcap_{i \in I} R_i\right)(x,y) = \bigwedge_{i \in I} R_i(x,y)$

Proposition 270 $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$;

Proof. $(R \cup S)^{-1}(x,y) = (R \cup S)(y,x) = R(y,x) \vee S(y,x)$
$= R^{-1}(x,y) \vee S^{-1}(x,y)$
$= \left(R^{-1} \cup S^{-1}\right)(x,y)$ ■

Proposition 271 $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$;

Proof. $(R \cap S)^{-1}(x,y) = (R \cap S)(y,x) = R(y,x) \wedge S(y,x)$
$= R^{-1}(x,y) \wedge S^{-1}(x,y)$
$= \left(R^{-1} \cap S^{-1}\right)(x,y)$ ■

Proposition 272 $(R^c)^{-1} = (R^{-1})^c$.

Proof. $(R^c)^{-1}(x,y) = R^c(y,x)$
$= 1 - R(y,x)$
$= 1 - R^{-1}(x,y)$
$= (R^{-1})^c(x,y)$ ■

A fuzzy relation also has the concept of (strong) α-cut. The crisp relation $R_\alpha = \{(x,y) \mid R(x,y) \geq \alpha\}$ for $\alpha \in [0,1]$ will be called the α-cut relation of R, and $\hat{R}_\alpha = \{(x,y) \mid R(x,y) > \alpha\}$ for $\alpha \in [0,1]$ will be called the strong α-cut relation of R.

Clearly, both an α-cut relation and a strong α-cut relation are crisp relations from X to Y. Naturally, (strong) α-cut relations have all the properties valid for (strong) α-cuts of a fuzzy set, e.g. $(R \cup S)_\alpha = R_\alpha \cup S_\alpha$,
$(R^c)_\alpha = (R_{1-\alpha})^c$, $R(x,y) = \bigvee_{\alpha \in [0.1]} (\alpha \wedge R_\alpha(x,y))$

Let R be a fuzzy relation from X to Y, where $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_m\}$. In this case, by letting $r_{ij} = R(x_i, y_j)$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$, the fuzzy relation R may be represented in the form of a matrix

$$\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & \ddots & & r_{2m} \\ \vdots & & \ddots & \vdots \\ r_{n1} & \cdots & \cdots & r_{nm} \end{pmatrix}$$

Thus, we can simply write we simply write $R = (r_{ij})_{n \times m}$

Example 273 *Given the universe of height $X = \{140, 150, 160, 170, 180\}$ in cm and the universe of weight $Y = \{40, 50, 60, 70, 80\}$ in kg, the relation between the height and weight of a person may be regarded as a fuzzy relation R which is expressed as:*

$$\begin{pmatrix} 1 & 0.8 & 0.2 & 0.1 & 0 \\ 0.8 & 1 & 0.8 & 0.2 & 0.1 \\ 0.2 & 0.8 & 1 & 0.8 & 0.2 \\ 0.1 & 0.2 & 0.8 & 1 & 0.8 \\ 0 & 0.1 & 0.2 & 0.8 & 1 \end{pmatrix}$$

In this case, $140R = 0.8/40 + 1/50 + 0.8/60 + 0.2/70 + 0.1/80$ *and* $R70 = 0.1/140 + 0.2/150 + 0.8/160 + 1/170 + 0.8/180$

Proposition 274 *Let $R = (r_{ij})_{n \times m}$ and $S = (s_{ij})_{n \times m}$. Then*
(1) $R \cup S = (r_{ij} \vee s_{ij})_{n \times m}$
(2) $R \cap S = (r_{ij} \vee s_{ij})_{n \times m}$
(3) $R^c = (1 - r_{ij})_{n \times m}$
(4) $R^{-1} = R^T$*, where* R^T *stands for the transpose of* R*.*

Proof. (1) $(R \cup S)(x_i, y_j) = R(x_i, y_j) \vee S(x_i, y_j) = (r_{ij} \vee s_{ij})$
$\implies R \cup S = (r_{ij} \vee s_{ij})_{n \times m}$

(2) $(R \cap S)(x_i, y_j) = R(x_i, y_j) \wedge S(x_i, y_j) = (r_{ij} \wedge s_{ij})$
$\Longrightarrow R \cap S = (r_{ij} \wedge s_{ij})_{n \times m} = (r_{ij} \wedge s_{ij})_{n \times m}$
(3) $R^c(x_i, y_j) = 1 - R(x_i, y_j) = 1 - r_{ij}$
$\Longrightarrow R^c = (1 - r_{ij})_{n \times m}$
(4) $R^{-1}(x_i, y_j) = R(y_j, x_i) = r_{ji}$
$\Longrightarrow R^{-1} = (r_{ji})_{n \times m}$
$\Longrightarrow R^{-1} = R^T$ ■

Definition 275 *(1) The complement R^c_η of R under η is defined by $\forall(x, y) \in X \times Y, R^c_\eta(x, y) = \eta(R(x, y))$.*

Definition 276 *(2) The union $R_1 \cup_S R_2$ of R_1 and R_2 under S is defined by $\forall(x, y) \in X \times Y,\ R_1 \cup_S R_2(x, y) = S(R_1(x, y), R_2(x, y))$.*

Definition 277 *(3) The intersection $R_1 \cap_T R_2$ of R_1 and R_2 under T is defined by $\forall(x, y) \in X \times Y\ (R_1 \cap_T R_2)(x, y) = T(R_1(x, y), R_2(x, y))$.*

2.3.1 Composition of Fuzzy Relations

Motivated by the characteristic function expression of the round composition of crisp relations, the round composition of two fuzzy relations is defined as follows.

Definition 278 *Let $R \in F(X \times Y)$, $S \in F(Y \times Z)$ and $T \in F(X \times Z)$ be three fuzzy relations. If $\forall(x, z) \in X \times Z$,*

$$T(x, z) = \bigvee_{y \in Y} (R(x, y) \wedge S(y, z)) = hgt([xR] \cap [Sz])$$

*then T is called **the (round) composition** of R and S, denoted by $R \circ S$.*

If R is a fuzzy relation on X, we employ R^2 to denote $R \circ R$ and define R^n (n is any positive integer greater than 1) recursively by $R^n = R^{n-1} \circ R$. In the case of finite universes, the composition can be readily performed by means of matrices. To illustrate this point, let $X = \{x_1, x_2, ..., x_l\}, Y = \{y_1, y_2, ..., y_m\}$ and $Z = \{z_1, z_2, ..., z_n\}$ and let $R = (r_{ij})_{l \times m}$, $S = (s_{ij})_{m \times n}$ and $T = (t_{ij})_{l \times n}$. By the definition of composition, $T = R \circ S$ means

$$T(x_i, z_k) = \bigvee_{y_j \in Y} (R(x_i, y_j) \wedge S(y_j, z_k)) = hgt([xR] \cap [Sz])$$

or equivalently $t_{ik} = \bigvee\limits_{j=1}^{m} (r_{ij} \wedge s_{jk})$ for $i = 1, 2, ..., l$ and $k = 1, 2, ..., n$

Example 279 *if* $R = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1 & 0 & 0.9 \end{pmatrix}$ *and* $S = \begin{pmatrix} 0.8 & 0.3 \\ 1 & 0 \\ 0.5 & 0.6 \end{pmatrix}$ *then* $R \circ S$

$$\begin{pmatrix} (0.3 \wedge 0.8) \vee (0.7 \wedge 0.1) \vee (0.2 \wedge 0.5) & (0.3 \wedge 0.3) \vee (0.7 \wedge 0) \vee (0.2 \wedge 0.6) \\ (1 \wedge 0.8) \vee (0 \wedge 0.1) \vee (0.9 \wedge 0.5) & (1 \wedge 0.3) \vee (0 \wedge 0) \vee (0.9 \wedge 0.6) \end{pmatrix} =$$

$$\begin{pmatrix} 0.3 & 0.3 \\ 0.8 & 0.6 \end{pmatrix}$$

Proposition 280 *The composition of fuzzy relations fulfills the following properties provided that the involved compositions are possible to perform.*

(1) $(R \circ S) \circ T = R \circ (S \circ T)$*;*

(2) $R \subseteq S$ *implies that* $R \circ T \subseteq S \circ T$ *and* $T \circ R \subseteq T \circ S$*, especially* $R \subseteq S$ *implies* $R^n \subseteq S^n$ *for any positive integer n;*

(3) $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$;

(4) $(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$ *and* $T \circ (R \cup S) = (T \circ R) \cup (T \circ S)$*;*

(5) $(\hat{R} \circ \hat{S})_\alpha = \hat{R}_\alpha \circ \hat{S}_\alpha$ *and if the involved universes are finite, then* $(R \circ S)_\alpha = R_\alpha \circ S_\alpha$

(6) $(R \cap S) \circ T \subseteq (R \circ T) \cap (S \circ T)$.

Proof. (1) Let $R \in F(X \times Y_1)$, $S \in F(Y_1 \times Y_2), T \in F(Y_2 \times Z)$. Then $\forall (x, z) \in X \times Z$

$[(R \circ S) \circ T](x, z) = \bigvee_{y_2 \in Y_2} [(R \circ S)(x, y_2) \wedge T(y_2, z)]$

$= \bigvee_{y_2 \in Y_2} \left[\left(\bigvee_{y_1 \in Y_1} R(x, y_1) \wedge S(y_1, y_2) \right) \wedge T(y_2, z) \right]$

$= \bigvee_{y_2 \in Y_2} \bigvee_{y_1 \in Y_1} (R(x, y_1) \wedge S(y_1, y_2) \wedge T(y_2, z))$

$= \bigvee_{y_1 \in Y_1} \left[R(x, y_1) \wedge \bigvee_{y_2 \in Y_2} (S(y_1, y_2) \wedge T(y_2, z)) \right]$

$= \bigvee_{y_1 \in Y_1} [R(x, y_1) \wedge (S \circ T)(y_1, z)]$

$= R \circ (S \circ T)(x, z)$

(2) Take $T \in F(Y \times Z)$ and $R, S \in F(X \times Y)$. Since $R(x, y) \leq S(x, y)$ $\forall (x, y) \in X \times Y$

$(R \circ T)(x, z) = \bigvee_{y \in Y} R(x, y) \wedge T(y, z)$

$\leq \bigvee_{y \in Y} S(x, y) \wedge T(y, z) = (S \circ T)(x, z)$

Next, take $T \in F(X \times Y)$ and $R, S \in F(Y \times Z)$. Since $R(y, z) \leq S(y, z)$ $\forall (y, z) \in Y \times Z$

Then, $(T \circ R)(x, z) = \bigvee_{y \in Y} R(y, z) \wedge T(x, y)$

$\leq \bigvee_{y \in Y} S(y, z) \wedge T(x, y)$

(3) Take $R \in F(X \times Y)$ and $S \in F(Y \times Z)$. Then, $R^{-1} \in F(Y \times X)$ and $S^{-1} \in F(Z \times Y)$

$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$;

Then, $(R \circ S)^{-1}(z, x) = (R \circ S)(x, z)$

$= \bigvee_{y \in Y} R(x, y) \wedge S(y, z)$

$= \bigvee_{y \in Y} S^{-1}(z, y) \wedge R^{-1}(y, x)$

$\left(S^{-1} \circ R^{-1}\right)(z, x)$

(4) Let $R, S \in F(X \times Y)$ and $T \in F(Y \times Z)$. For $(x, z) \in X \times Z$,
$[(R \cup S) \circ T](x, z)$
$= \bigvee_{y \in Y} (R \cup S)(x, y) \wedge T(y, z)$
$= \bigvee_{y \in Y} (R(x, y) \vee S(x, y)) \wedge T(y, z)$
$= \bigvee_{y \in Y} [(R(x, y) \wedge T(y, z)) \vee (S(x, y) \wedge T(y, z))]$
$= \left[\bigvee_{y \in Y} (R(x, y) \wedge T(y, z))\right] \vee \left[\bigvee_{y \in Y} (S(x, y) \wedge T(y, z))\right]$
$= (R \circ T)(x, z) \vee (S \circ T)(x, z)$
$= [(R \circ T) \cup (S \circ T)](x, z)$
(5) Let $R \in F(X \times Y)$ and $S \in F(Y \times Z)$. Then $(x, z) \in (\hat{R} \circ \hat{S})_\alpha$
$\iff (R \circ S)(x, z) > \alpha$
$\iff \bigvee_{y \in Y} (R(x, y) \wedge S(y, z)) > \alpha$
$\iff \exists y \in Y,\ R(x, y) \wedge S(y, z) > \alpha$
$\iff \exists y \in Y,\ R(x, y) > \alpha$ and $S(y, z) > \alpha$
$\iff \exists y \in Y,\ (x, y) \in \hat{R}_\alpha$ and $(y, z) \in \hat{S}_\alpha$
$\iff (x, z) \in \hat{R}_\alpha \circ \hat{S}_\alpha$
(6) Take $R, S \in F(X \times Y)$ and $T \in F(Y \times Z)$
Then, for any $(x, z) \in X \times Z$
$((R \cap S) \circ T)(x, z)$
$= \bigvee_{y \in Y} [(R \cap S)(x, y) \wedge T(y, z)]$
$= \bigvee_{y \in Y} R(x, y) \wedge S(x, y) \wedge T(y, z)$
$= \bigvee_{y \in Y} (R(x, y) \wedge T(y, z)) \wedge (S(x, y) \wedge T(y, z))$
$\leq \left[\bigvee_{y \in Y} (R(x, y) \wedge T(y, z))\right] \wedge \left[\bigvee_{y \in Y} (S(x, y) \wedge T(y, z))\right]$
$[(R \circ T) \cap (S \circ T)](x, z)$ ■

2.3.2 Fuzzy Equivalence

Definition 281 *If $R(x, x) = 1\ \forall x \in X$, then R is called a* ***reflexive (fuzzy) relation****.*

If X is finite and $R = (r_{ij})_{n \times n}$, reflexivity implies that $r_{ii} = 1 (i = 1, 2, ..., n)$ and vice versa. As a result, we can observe the numbers on the principal diagonal of R to judge whether R is reflexive or not.

Proposition 282 *R is reflexive iff $\forall \alpha \in [0, 1]$, R_α is reflexive.*

Proof. If R is reflexive, then $\forall \alpha \in [0, 1]$, $R(x, x) = 1 \geq \alpha$. Hence $(x, x) \in R_\alpha$, viz. R_α is reflexive.

Conversely, assume that $\forall \alpha \in [0, 1]$, R_α is reflexive. Particularly, R_1 is reflexive. Hence $\forall x \in X$,

$(x, x) \in R_1$, or $R(x, x) = 1$. ■

It follows from that R is reflexive iff R_1 (1-cut relation of R) is reflexive.

Definition 283 *If* $\forall x, y \in X$, $R(x, y) = R(y, x)$, *then* R *is called a* ***symmetric (fuzzy) relation****.*

Obviously, R is symmetric iff $R = R^{-1}$. We know that $R^{-1} = R^T$ in the case of finite universes. Hence R is a symmetric relation iff R as a matrix is symmetric in this case.

Proposition 284 *R is symmetric iff* $\forall \alpha \in [0, 1]$, R_α *is a symmetric relation.*

Proof. If R is symmetric and $(x, y) \in R_\alpha$, then $R(y, x) = R(x, y) \geq \alpha$. Hence $(y, x) \in R_\alpha$, which proves the symmetry of R_α. Conversely, assume that $\forall \alpha \in [0, 1]$, R_α is symmetric. For any $x, y \in X$, take $\alpha = R(x, y)$. Then $(x, y) \in R_\alpha$ and hence $(y, x) \in R_\alpha$ due to the symmetry of R_α. Therefore $R(y, x) \geq \alpha = R(x, y)$.

Next, $(x, y) \in R_\alpha$ and hence $(y, x) \in R_\alpha$ implies $R(x, y) \geq \alpha$ and $R(y, x) \geq \alpha$. We can take $R(y, x) = \alpha$ so that $R(x, y) \geq R(y, x)$. Combining the two inequalities yields $R(x, y) = R(y, x)$. ■

Definition 285 *If* $R \supseteq R^2$, *then* R *is said to be a* ***transitive (fuzzy) relation.***

Proposition 286 *R is transitive iff* $\forall x, y, z \in X$, $R(x, z) \geq R(x, y) \wedge R(y, z)$.

Proof. R is transitive

$\iff R \supseteq R^2$

$\iff \forall x, z \in X,\ R(x, z) \geq R^2(x, y)$

$\iff \forall x, y, z \in X,\ R(x, z) \geq R(x, y) \wedge R(y, z)$ ■

If $X = \{x_1, x_2, ..., x_n\}$ is finite and $R = (r_{ij})_{n \times n}$, then R is transitive iff $R(x_i, x_k) \geq R(x_i, x_j) \wedge R(x_j, x_k)$, i.e. $r_{ik} \geq r_{ij} \wedge r_{jk}$ for $i, j, k = 1, 2, ..., n$.

Proposition 287 *R is transitive iff* $\forall \alpha \in [0, 1]$, R_α *is transitive.*

Proof. ($\Longrightarrow$) Let $(x, y), (y, z) \in R_\alpha$ for any fixed $\alpha \in [0, 1]$. It follows that $R(x, y) \geq \alpha$ and $R(y, z) \geq \alpha$. Then, $R(x, y) \wedge R(y, z) = \alpha \leq R(x, z) \Longrightarrow (x, z) \in R_\alpha$

($\Longleftarrow$) We prove $\forall x, y, z \in X$, $R(x, z) \geq R(x, y) \wedge R(y, z)$. By letting $R(y, x) \wedge R(y, z) = \alpha$, we have $R(x, y) \geq \alpha$ and $R(y, z) \geq \alpha$ so that $(x, y) \in R_\alpha$ and $(y, z) \in R_\alpha$. Hence $R(x, z) \geq R(y, x) \wedge R(y, z) = \alpha$ since R_α is transitive. ■

Definition 288 *If R is reflexive, symmetric and transitive, then R is called a* ***fuzzy equivalence relation.***

Proposition 289 *R is a fuzzy equivalence relation iff* $\forall \alpha \in [0, 1]$, R_α *is an equivalence relation.*

Proof. Direct consequence of previous three propositions ■

We know that a crisp equivalence relation determines a partition of X. So every R_α determines a partition of X if R is a fuzzy equivalence relation. For example, let R be a fuzzy relation on $X = \{x_1, x_2, x_3, x_4, x_5\}$, defined by

$$R = \begin{pmatrix} 1 & 0.4 & 0.8 & 0.5 & 0.5 \\ 0.4 & 1 & 0.4 & 0.4 & 0.4 \\ 0.8 & 0.4 & 1 & 0.5 & 0.5 \\ 0.5 & 0.4 & 0.5 & 1 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.6 & 1 \end{pmatrix}$$

Apparently, R is a reflexive because the diagonal elements are 1. R is also symmetric fuzzy relation because $R^T = R$. In addition, it is easily checked that $R^2 = R$, making R transitive. Thus R is a fuzzy equivalence relation.

For $0.8 < \alpha \leq 1$, $R_\alpha = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_5, x_5)\}$, the partition of X determined by R_α is $\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}$.

Similarly, for $0.6 < \alpha \leq 0.8$, $R_\alpha = \{(x_1, x_3), (x_3, x_1)\}$ the partition of X determined by R_α is $\{x_1, x_3\}, \{x_2\}, \{x_4\}, \{x_5\}$.

For $0.5 < \alpha \leq 0.6$, $R_\alpha = \{(x_4, x_5), (x_5, x_4)\}$ the partition of X determined by R_α is $\{x_1, x_3\}, \{x_2\}, \{x_4, x_5\}$.

For $0.4 < \alpha \leq 0.5$, $R_\alpha = \{(x_1, x_4), (x_4, x_1), (x_3, x_4), (x_3, x_5), (x_5, x_3), (x_4, x_3)\}$ the partition of X determined by R_α is

$\{x_1, x_3, x_4, x_5\}, \{x_2\}$.

If $\alpha \leq 0.4$, the elements in X cannot be partitioned by R_α. Clearly, the partition determined by R_α becomes increasingly refined as the α increases.

Definition 290 *Let R be a fuzzy equivalence relation on X. A fuzzy set $[a]_R$ for $a \in X$ defined by:*

$\forall x, \in X$, $[a]_R(x) = R(a, x)$ is called the fuzzy equivalence class of a by R. The set $X/R = \{[a]_R \mid a \in X\}$ of all fuzzy equivalence classes is called the fuzzy quotient set of X by R.

Example 291 *Let $X = \{a, b, c\}$ and R be a fuzzy equivalence relation on X defined by* $R = \begin{pmatrix} 1 & 1 & 0.7 \\ 1 & 1 & 0.7 \\ 0.7 & 0.7 & 1 \end{pmatrix}$. *Then $[a]_R = 1/a + 1/b + 0.7/c$; $[b]_R = 1/a + 1/b + 0.7/c$ and $[c]_R = 0.7/a + 0.7/b + 1/c$. The fuzzy quotient set of X by R is $X/R = \{[a]_R, [c]_R\}$.*

We know that $[a]_R = [b]_R$ iff aRb in the crisp case. The following is a fuzzy counterpart of this result.

Proposition 292 *If R is a fuzzy equivalence relation, then $[a]_R = [b]_R$ iff $R(a, b) = 1$.*

Proof. $(\Longrightarrow)$ $R(a, b) = [a]_R(b) = [b]_R(b) = R(b, b) = 1$.

$(\Longleftarrow)$ If $R(a, b) = 1$, then $\forall x \in X$, $[a]_R(x) = R(a, x) \geq R(a, b) \wedge R(b, x) = R(b, x) = [b]_R(x)$

since R is transitive. Similarly, $[b]_R(x) = R(b,x) \geq R(b,a) \wedge R(a,x) = R(a,x) = [a]_R(x)$ we have $[b]_R(x) \geq [a]_R(x)$. Consequently, $[a]_R = [b]_R$. ■

Unlike the crisp case, the intersection of two distinct fuzzy equivalence classes may be not empty. For instance, $[a]_R \cap [c]_R = 0.7/a + 0.7/b + 0.7/c$, which is a non-empty set in the above example. We have the following weaker result instead.

Proposition 293 *If* $[a]_R = [b]_R$*, then* $hgt([a]_R \cap [b]_R) < 1$.

Proof. If $[a]_R \neq [b]_R$ and $hgt([a]_R \cap [b]_R) = 1$, then due to the transitivity of R, we have

$R(a,b)$

$\geq \bigvee_{x \in X} (R(a,x) \wedge R(x,b))$

$= \bigvee_{x \in X} ([a]_R(x) \wedge [b]_R(x))$

$hgt([a]_R \cap [b]_R) = 1$ which contradicts $[a]_R = [b]_R$ iff $R(a,b) = 1$. ■

3 Fuzzy Analysis and Algebra

As known to us, the theory of classical sets is the foundation on which modern mathematics rests. When sets are fuzzified, some traditional pure mathematical branches are accordingly generalized. In this chapter, we introduce three well-developed fuzzified mathematical areas briefly to have a glance at how a pure mathematical theory can be fuzzified. The three areas are (i) fuzzy measures and fuzzy integrals (ii) fuzzy algebraic structures including fuzzy groups, fuzzy rings and fuzzy fields (iii) fuzzy topology. This chapter will be mainly for authors with the elementary knowledge of the corresponding classical mathematical branches and it will supply them with basic materials for further reading or research.

3.1 Fuzzy Measures

In mathematical analysis and in probability theory, a σ-algebra (also sigma-algebra, σ-field, sigma-field) on a set X is a collection of subsets of X that is closed under countably many set operations (complement, union and intersection). On the other hand, an algebra is only required to be closed under finitely many set operations. That is, a σ-algebra is an algebra of sets, completed to include countably infinite operations.

More rigorously,

Definition 294 *Let X be some set. Then a subset $\Sigma \subseteq \mathcal{P}(X)$ is called a* ***σ-algebra*** *if it satisfies the following three properties:*

Σ is non-empty: There is at least one $A \subset X$ in Σ.

Σ is closed under complementation: If A is in Σ, then so is its complement, $X \backslash A$.

Σ is closed under countable unions: If $A_1, A_2, A_3, ...$ are in Σ, then so is $A = \bigcup_i A_i$

The main use of σ-algebras is in the definition of measures; specifically, the collection of those subsets for which a given measure is defined is necessarily a σ-algebra. This concept is important in mathematical analysis as the foundation for Lebesgue integration

If $X = \{a, b, c, d\}$, one possible σ-algebra on X is $\Sigma = \{\varnothing, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$, where $\varnothing$ is the empty set. However, a finite algebra is always a σ-algebra. If $\{A_1, A_2, A_3, \ldots\}$ is a countable partition of X then the collection of all unions of sets in the partition (including the empty set) is a σ-algebra.

Proposition 295 *Let (X, Σ) be a σ-algebra. Then, the countable intersection of elements of Σ is in Σ*

Proof. $\bigcup_i A_i \in \Sigma$ and $A_i^c \in \Sigma\ \forall i \implies \left(\bigcup_i A_i\right)^c = \bigcap_i A_i \in \Sigma$ ■

Definition 296 *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra and $A, B \in \mathcal{A}$,. If a mapping $g : \mathcal{A} \longrightarrow [0, 1]$ satisfies*

(1) boundedness: $g(\varnothing) = 0$ and $g(X) = 1$

(2) monotonicity: $A \subseteq B$ implies $g(A) \leq g(B)$

(3) continuity: $A_n \uparrow$(or $\downarrow$) A (read $A_n \longrightarrow A$ monotonically) implies that $\lim_{n\to\infty} g(A_n) = g(A)$,

then g is called a ***fuzzy measure.***

(X, A) and (X, A, g) are called a fuzzy measurable space and a fuzzy measure space respectively.

Sugeno made the following interpretation: g(A) measures the certainty degree to which a generic element x is in A. If A is empty, x is certainly not in A. If A is the whole set, x is certainly in it. When $A \subseteq B$, the certainty degree to which x is in A is of course less than the certainty degree to which x is in B.

Example 297 *For any $A \in \mathcal{P}(X)$ and $A, B \in \mathcal{A}$, the Dirac measure centered in $x_0 \in X$ assumes the form*

$$g(A) = A(x_0) = \begin{matrix} 1 & x_0 \in A \\ 0 & x_0 \notin A \end{matrix}$$

where x_0 is a fixed element in X. To show that this is indeed a fuzzy measure, $g(\varnothing) = \varnothing(x_0) = 0$ since $x_0 \notin \varnothing$. Next, $g(X) = X(x_0) = 1$ since $x \in X$ by default. For (2), let $A \subseteq B$. If A is empty, then (2) trivially holds. Assume A is non-empty. Then, $g(A) = A(x_0) = 1$ if $x_0 \in A \implies x_0 \in B \implies g(B) = 1$. Furthermore, if $g(A) = A(x_0) = 0$. Then, $x_0 \notin A$. Since $0 \leq g\,(U) \in \mathcal{P}(X)$ and in particular for $g(B)$. Thus, in all cases, $g(A) \leq g(B)$.

Finally, let $A_n \uparrow$(or $\downarrow$) A. If $A_n\,(x_0) = 0\ \forall n$, then $A\,(x_0) = 0$. If $A_n\,(x_0) = 1$ $\forall n$, then $A\,(x_0) = 1$ for some finite starting n, then $A\,(x_0) = 1$ or $A\,(x_0) = 0$ (this case is confusing). In all cases, $\lim_{n\to\infty} g(A_n) = g(A)$,

Proposition 298 *If g is a fuzzy measure on the measurable space (X, A) and $A, B \in \mathcal{A}$, then*

(1) $g(A \cup B) \geq g(A) \vee g(B)$;

(2) $g(A \cap B) \leq g(A) \wedge g(B)$.

Proof. Clearly, we have $A \cup B \supseteq A$ and $A \cup B \supseteq B$. Since g is monotonic, therefore 1 holds

Next, $A \cap B \subseteq A$ and $A \cap B \subseteq B$ so that 2 holds ■

More generally, $g\left(\bigcup_i A_i\right) \geq \bigvee_i g\left(A_i\right)$ and $g\left(\bigcap_i A_i\right) \geq \bigwedge_i g\left(A_i\right)$

Definition 299 *If a mapping $g_\lambda : A \longrightarrow [0,1]$ depending on a parameter λ $(\lambda > -1)$ satisfies that*

(1) $g_\lambda(X) = 1$,

(2) $g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)$ whenever $A \cap B = \varnothing$,

(3) $A_n \uparrow$(or $\downarrow$) A implies that $\lim_{n\to\infty} g_\lambda\left(A_n\right) = g_\lambda(A)$,

*then g_λ is called a **λ-fuzzy measure** or a g_λ **measure.***

Proposition 300 *Each g_λ measure is a fuzzy measure.*

Proof. Since $X \cap \varnothing = \varnothing$ and $X \cup \varnothing = X$, then we can apply (2) of g_λ measure

$g_\lambda(X \cup \varnothing) = g_\lambda(X) = g_\lambda(X) + g_\lambda(\varnothing) + \lambda g_\lambda(X) g_\lambda(\varnothing)$

or $g_\lambda(\varnothing) + \lambda g_\lambda(\varnothing) = 0$

or $g_\lambda(\varnothing)\left(1 + \lambda\right) = 0$

Since $\lambda > -1$, we have $\lambda + 1 > 0$ ir $\lambda + 1 \neq 0$ so that $g_\lambda(\varnothing) = 0$. From (1) of g_λ measure and $g_\lambda(\varnothing) = 0$, this satisfies (1) of fuzzy measure,

Assume that $A \subseteq B$. Then $A \cup (B - A) = A \cup B = B$, together with $A \cap (B - A) = \varnothing$ leads to

$g_\lambda(B) = g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)$

or $g_\lambda(A) + \lambda g_\lambda(A) g_\lambda(B) = 0$

or $g_\lambda(A)\left(1 + \lambda g_\lambda(B)\right) \geq g_\lambda(A)$

(3) already holds ■

Proposition 301 *Each g_λ measure satisfies the following properties.*

(1) $g_\lambda\left(A^c\right) = \frac{1 - g_\lambda(A^c)}{1 + \lambda g_\lambda(A)}$

(2) If $A \supseteq B$, then $g_\lambda(A - B) = \frac{g_\lambda(A) - g_\lambda(B)}{1 + \lambda g_\lambda(B)}$

(3) If $A_i \cap A_j = \varnothing$ for $i \neq j$, then $g_\lambda\left(\bigcup_n A_n\right) = \frac{1}{\lambda}\left(\prod_n 1 + \lambda g_\lambda\left(A_n\right)\right) - \frac{1}{\lambda}$

Proof. (1) From $A \cap A^c = \varnothing$, we have $1 = g\left(X\right) = g_\lambda(A \cup A^c) = g_\lambda(A) + g_\lambda(A^c) + \lambda g_\lambda(A) g_\lambda(A^c)$ or

$1 = g_\lambda(A)\left(1 + \lambda g_\lambda(A^c)\right) + g_\lambda(A^c)$

or $1 - g_\lambda(A^c) = g_\lambda(A)\left(1 + \lambda g_\lambda(A^c)\right)$

or $g_\lambda\left(A^c\right) = \frac{1 - g_\lambda(A^c)}{1 + \lambda g_\lambda(A)}$

(2) Suppose $A \supseteq B$, Then, $A = B \cup (A - B)$ and $B \cap (A - B) = \varnothing$. Therefore, we use (2) to get

$g_\lambda(A) = g_\lambda(B \cup (A-B)) = g_\lambda(B) + g_\lambda(A-B) + \lambda g_\lambda(B) g_\lambda(A-B)$
or $g_\lambda(A) - g_\lambda(B) = g_\lambda(A-B)\,(1 + \lambda g_\lambda(B))$
or $\frac{g_\lambda(A)-g_\lambda(B)}{1+\lambda g_\lambda(B)} = g_\lambda(A-B)$

(3) Assume From $A_1 \cap A_2 = \varnothing$. Then, $g_\lambda(A_1 \cup A_2) = g_\lambda(A_1) + g_\lambda(B_2) + \lambda g_\lambda(A_1) g_\lambda(B_2)$
or $g_\lambda(A_1 \cup A_2) = g_\lambda(A_1) + g_\lambda(A_2)\,(1 + \lambda g_\lambda(A_1))$
or $g_\lambda(A_1 \cup A_2) = \frac{1}{\lambda} + g_\lambda(A_1) + \lambda g_\lambda(A_2)\left(\frac{1}{\lambda} + g_\lambda(A_1)\right) - \frac{1}{\lambda}$
or $g_\lambda(A_1 \cup A_2) = \left(\frac{1}{\lambda} + g_\lambda(A_1)\right)(1 + \lambda g_\lambda(A_2)) - \frac{1}{\lambda}$
or $g_\lambda(A_1 \cup A_2) = \frac{1}{\lambda}(1 + \lambda g_\lambda(A_1))(1 + \lambda g_\lambda(A_2)) - \frac{1}{\lambda}$

Thus, (3) is valid for $n = 2$

Let $g_\lambda\left(\bigcup_i^n A_i\right) = \frac{1}{\lambda}\left(\prod_i^n 1 + \lambda g_\lambda(A_i)\right) - \frac{1}{\lambda}$ be true if $A_i \cap A_j = \varnothing$ for $i \neq j$

Then, $g_\lambda\left(\bigcup_i^{n+1} A_i\right) = g_\lambda\left(\bigcup_i^n A_i\right) + g_\lambda(A_{n+1}) + \lambda g_\lambda\left(\bigcup_i^n A_i\right) g_\lambda(A_{n+1})$

or $g_\lambda\left(\bigcup_i^{n+1} A_i\right) = \frac{1}{\lambda}\left(\prod_i^n 1 + \lambda g_\lambda(A_i)\right) - \frac{1}{\lambda} + g_\lambda(A_{n+1}) + \lambda\left[\frac{1}{\lambda}\left(\prod_i^n 1 + \lambda g_\lambda(A_i)\right) - \frac{1}{\lambda}\right] g_\lambda(A_{n+1})$

or $g_\lambda\left(\bigcup_i^{n+1} A_i\right) = \frac{1}{\lambda}\left(\prod_i^n 1 + \lambda g_\lambda(A_i)\right) - \frac{1}{\lambda} + \left(\prod_i^n (1 + \lambda g_\lambda(A_i))\right) g_\lambda(A_{n+1})$

or $g_\lambda\left(\bigcup_i^{n+1} A_i\right) = \left(\prod_i^n 1 + \lambda g_\lambda(A_i)\right)\left(\frac{1}{\lambda} + g_\lambda(A_{n+1})\right) - \frac{1}{\lambda}$

or $g_\lambda\left(\bigcup_i^{n+1} A_i\right) = \frac{1}{\lambda}\left(\prod_i^n 1 + \lambda g_\lambda(A_i)\right)(1 + \lambda g_\lambda(A_{n+1})) - \frac{1}{\lambda}$

or $g_\lambda\left(\bigcup_i^{n+1} A_i\right) = \frac{1}{\lambda}\left(\prod_i^{n+1} 1 + \lambda g_\lambda(A_i)\right) - \frac{1}{\lambda}$

Hence $g_\lambda\left(\bigcup_i^k A_i\right) = \frac{1}{\lambda}\left(\prod_i^k 1 + \lambda g_\lambda(A_i)\right) - \frac{1}{\lambda}$ is valid for all k.

Then, $\lim_{k\to\infty} g_\lambda\left(\bigcup_i^k A_i\right) = \lim_{k\to\infty} \frac{1}{\lambda}\left(\prod_i^k 1 + \lambda g_\lambda(A_i)\right) - \frac{1}{\lambda}$

or $g_\lambda\left(\bigcup_n A_n\right) = \frac{1}{\lambda}\left(\prod_n 1 + \lambda g_\lambda(A_n)\right) - \frac{1}{\lambda}$ because g_λ is continuous ■

Proposition 302 *For $A, B \in \mathcal{A}$, $g_\lambda(A \cup B) = \frac{g_\lambda(A)+g_\lambda(B)-\lambda g_\lambda(A) g_\lambda(B)}{1+\lambda g_\lambda(A\cap B)}$*

Proof. On the one hand, $g_\lambda(A \cup B) = g_\lambda((A \cup B) \cap X)$
$= g_\lambda((A \cup B) \cap (A \cup A^c))$
$= g_\lambda(A \cup (B \cap A^c))$
$= g_\lambda(A \cup (B - A))$

Since $A \cap (B - A) = \varnothing$, we use (2) to get
$g_\lambda(A \cup B) = g_\lambda(A \cup (B-A)) = g_\lambda(A) + g_\lambda(B-A)) + \lambda g_\lambda(A) g_\lambda(B-A)$

On the other hand, $g_\lambda(B) = g_\lambda(B \cap X)$
$= g_\lambda(B \cap (A \cup A^c))$
$= g_\lambda((B \cap A) \cup (B \cap A^c))$

Since $(B \cap A) \cap (B \cap A^c) = B \cap A \cap A^c = \varnothing$, we can again use (2) to get

$g_\lambda(B) = g_\lambda\left((B\cap A)\cup(B-A)\right) = g_\lambda(B\cap A)+g_\lambda(B-A)+\lambda g_\lambda(B\cap A)g_\lambda(B-A)$

or $g_\lambda(B) - g_\lambda(B\cap A) = g_\lambda(B-A) + \lambda g_\lambda(B\cap A)g_\lambda(B-A)$

or $g_\lambda(B) - g_\lambda(B\cap A) = g_\lambda(B-A)\left(1+\lambda g_\lambda(B\cap A)\right)$

or $\frac{g_\lambda(B)-g_\lambda(B\cap A)}{1+\lambda g_\lambda(B\cap A)} = g_\lambda(B-A)$

Putting this in the previous equality, we get

$g_\lambda(A\cup B) = g_\lambda(A) + \frac{g_\lambda(B)-g_\lambda(B\cap A)}{1+\lambda g_\lambda(B\cap A)} + \lambda g_\lambda(A)\frac{g_\lambda(B)-g_\lambda(B\cap A)}{1+\lambda g_\lambda(B\cap A)}$

$\frac{(1+\lambda g_\lambda(A\cap B))g_\lambda(A)+\lambda g_\lambda(A)g_\lambda(B\cap A)+g_\lambda(B)-g_\lambda(B\cap A)+\lambda g_\lambda(A)g_\lambda(B)-\lambda g_\lambda(A)g_\lambda(B\cap A)}{1+\lambda g_\lambda(A\cap B)}$

$g_\lambda(A\cup B) = \frac{g_\lambda(A)+g_\lambda(B)-g_\lambda(B\cap A)+\lambda g_\lambda(A)g_\lambda(B)}{1+\lambda g_\lambda(A\cap B)}$ ■

Example 303 *Let $X = \{x_1, x_2, ..., x_n\}$ and $A = \mathcal{P}(X)$. If $g_i \in [0,1]$ for $i = 1, 2, ..., n$ satisfies $\prod_i^n (1+\lambda g_i) = 1+\lambda$ then g_λ defined by $\forall A \in \mathcal{A}$, $g_\lambda(A) = \frac{1}{\lambda}\prod_{x_i\in A}^n (1+\lambda g_i) - \frac{1}{\lambda}$ is a $\lambda-$fuzzy measure. Conversely, if g_λ is a $\lambda-$fuzzy measure, then the equalities hold for $g_\lambda(\{x_i\})$ for $i = 1, 2, ..., n$*

Proof. Assume that the equalities are satisfied. Then $g_\lambda(X) = \frac{1}{\lambda}\prod_{x_i\in X}^n (1+\lambda g_i) - \frac{1}{\lambda}$

$= \frac{1}{\lambda}\prod_i^n (1+\lambda g_i) - \frac{1}{\lambda} = \frac{1}{\lambda}(1+\lambda) - \frac{1}{\lambda} = 1$

Suppose that $A\cap B = \varnothing$. Write $a = \prod_{x_i\in A}^n (1+\lambda g_i)$ and $b = \prod_{x_i\in B}^n (1+\lambda g_i)$. Then,

$g_\lambda(A\cup B) = \frac{1}{\lambda}\prod_{x_i\in A\cup B}^n (1+\lambda g_i) - \frac{1}{\lambda}$

$= \frac{1}{\lambda}\left(\prod_{x_i\in A}^n (1+\lambda g_i)\right)\left(\prod_{x_i\in B}^n (1+\lambda g_i)\right) - \frac{1}{\lambda}$

$= \frac{1}{\lambda}(ab-1)$

$= \frac{1}{\lambda}ab - \frac{1}{\lambda} + \frac{1}{\lambda}a - \frac{1}{\lambda} + \frac{1}{\lambda} - \frac{1}{\lambda}a + \frac{1}{\lambda}b - \frac{1}{\lambda}b$

$= \frac{1}{\lambda}(a-1) + \frac{1}{\lambda}(b-1) + \frac{1}{\lambda}ab - \frac{1}{\lambda}a - \frac{1}{\lambda}b + \frac{1}{\lambda}b$

$= \frac{1}{\lambda}(a-1) + \frac{1}{\lambda}(b-1) + \frac{1}{\lambda}ab - \frac{1}{\lambda}a - \frac{1}{\lambda}b + \frac{1}{\lambda}b$

$= \frac{1}{\lambda}(a-1) + \frac{1}{\lambda}(b-1) + \lambda\frac{1}{\lambda}(a-1)\frac{1}{\lambda}(b-1)$

$= g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A)g_\lambda(B)$

Since X is a finite set, the continuity requirement is automatically satisfied. Thus g_λ is a λ-fuzzy measure.

Conversely, assume that g_λ is a λ-fuzzy measure.

$g_\lambda(\{x_1, x_2\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_2\}) + \lambda g_\lambda(\{x_1\})g_\lambda(\{x_2\})$

$= g_1 + g_2 + \lambda g_1 g_2 = \frac{1}{\lambda}(1+\lambda g_1)\left[(1+\lambda g_2) - 1\right]$

Hence the equality holds for $A = \{x_1, x_2\}$ for $n = 2$. Assume the equality holds for some k.

$g_\lambda\left(\bigcup_{i=1}^k x_i\right) = g_\lambda(\{x_k\}) + g_\lambda\left(\bigcup_{i=1}^{k-1} x_i\right) + \lambda g_\lambda(\{x_k\})g_\lambda\left(\bigcup_{i=1}^{k-1} x_i\right)$

Applying mathematical induction, we can prove that the equality is valid for all n. Observe that $g_\lambda(X) = 1$. By the same equality, $\frac{1}{\lambda}\left(\prod_i^n (1+\lambda g_i) - 1\right) = 1$, we get the second. ■

3.2 Fuzzy Algebra

In this section, we merely introduce the fuzzification of some main notions in abstract algebra including groups, normal groups, rings and ideals

3.2.1 Fuzzy Group

Definition 304 *A fuzzy subset A on G is called a **fuzzy subgroup** of G if it satisfies the following conditions:*
(1) $A(xy) \geq A(x) \wedge A(y)$ for any $x, y \in G$ and
(2) $A(x^{-1}) \geq A(x)$ for any $x \in G$.

As we know, a subset A of group G is a subgroup of G iff G satisfies that (1) $x, y \in A$ implies $xy \in A$ and (2) $x \in A$ implies $x^{-1} \in A$. The two inequalities in the above definition are just the fuzzification of these conditions.

Proposition 305 *Let A be a fuzzy subgroup of G. For any $x \in G$,*
(1) $A(x) \leq A(e)$,
(2) $A(x^{-1}) = A(x)$,
(3) $A(x^n) \geq A(x)$, where n is an arbitrary integer.

Proof. (1) $A(e) = A(xx^{-1}) \geq A(x) \wedge A(x^{-1}) \geq A(x) \wedge A(x) = A(x)$
(2) $A(x) = A\left(\left(x^{-1}\right)^{-1}\right) \geq A\left(x^{-1}\right)$
(3) Holds for $n = 2$. Assume it holds for k. Then, $A(x^{k+1}) \geq A(x) \wedge A\left(x^k\right) \geq A(x) \wedge A(x) = A(x)$ ■

Proposition 306 *Let $A \in F(G)$. Then A is a fuzzy subgroup of G iff $A(xy^{-1}) \geq A(x) \wedge A(y)$ holds for any $x, y \in G$.*

Proof. If A is a fuzzy subgroup of G, then $A(xy^{-1}) \geq A(x) \wedge A(y^{-1}) = A(x) \wedge A(y)$.
Conversely, suppose $A(xy^{-1}) \geq A(x) \wedge A(y)$ holds for any $x, y \in G$. Then for any $x \in G$,
$A(e) = A(xx^{-1}) \geq A(x) \wedge A(x) = A(x)$
i.e. $A(x) \leq A(e)$. Thus, for any $x \in G$, $A(x^{-1}) = A(ex^{-1}) \geq A(e) \wedge A(x) = A(x)$.
Meanwhile, for any $x, y \in G$, $A(xy) = A\left(x\left(y^{-1}\right)^{-1}\right) \geq A(x) \wedge A(y^{-1}) \geq A(x) \wedge A(y)$. Therefore A is a fuzzy subgroup of G. ■

Proposition 307 *A is a fuzzy subgroup of G iff A_α is a subgroup of G for every $\alpha \in \mathcal{R}(G)$.*

Proof. Suppose that A is a fuzzy subgroup of G and $\alpha \in \mathcal{R}(G)$. Then $\exists x$ such that $A(x) = \alpha$ so that $A_\alpha \neq \varnothing$. Let $x, y \in A_\alpha$, i.e. $A(x) \geq \alpha$ and $A(y) \geq \alpha$. Hence,

$A(xy^{-1}) \geq A(x) \wedge A(y^{-1}) = A(x) \wedge A(y) \geq \alpha$, and thus $xy^{-1} \in A_\alpha$.

Conversely, suppose A_α is a subgroup of G for every $\alpha \in A(G)$. For any $x, y \in G$, let $\alpha = A(x) \wedge A(y) \in A(G)$. Then $A(x) \geq \alpha$ and $A(y) \geq \alpha$, i.e. $x \in A_\alpha$ and $y \in A_\alpha$. Hence, $xy^{-1} \in A_\alpha$ since A_α is a subgroup of G. Consequently, $A(xy^{-1}) \geq \alpha = A(x) \wedge A(y)$. ■

Particularly, $A_{A(e)} = \{x \mid A(x) = A(e)\}$ is a subgroup of G if A is a fuzzy subgroup of G. We shall denote this subgroup by A^*

The binary multiplicative operation in G can be extended to $F(G)$ using the Zadeh's extension principle. Let $A, B \in F(G)$. Then $A \circ B$ is defined by: for any $z \in G$, $(A \circ B)(z) = \bigvee_{z=xy} (A(x) \wedge B(y))$. In addition, for every $A \in F(G)$, we shall define $A^{-1} \in F(G)$ by: for any $x \in G$, $A^{-1}(x) = A\left(x^{-1}\right)$. With these notions, we present an equivalent

statement of a fuzzy subgroup

Proposition 308 *Let $A \in F(G)$.Then A is a fuzzy subgroup of G iff $A \circ A^{-1} = A$.*

Proof. If A is a fuzzy subgroup of G, then for any $z \in G$, $(A \circ A^{-1})(z) = \bigvee_{z=xy} (A(x) \wedge A^{-1}(y))$

$= \bigvee_{z=xy} (A(x) \wedge A(y^{-1}))$

$= \bigvee_{z=xy} (A(x) \wedge A(y))$

$\leq \bigvee_{z=xy} A(xy) = A(z)$. Hence $A \circ A^{-1} \subseteq A$. Meanwhile, for any $z \in G$, $(A \circ A^{-1})(z) = \bigvee_{z=xy} (A(x) \wedge A^{-1}(y)) \geq A(z) \wedge A(e)$. Thus $A \circ A^{-1} \supseteq A$. Consequently, $A \circ A^{-1} = A$.

Conversely, suppose $A \circ A^{-1} = A$.Then, for any $x, y \in G$, $A(xy^{-1}) = \left(A \circ A^{-1}\right)\left(xy^{-1}\right) \geq A(x) \wedge A^{-1}(y^{-1}) = A(x) \wedge A(y)$. ■

Proposition 309 *Let A be a fuzzy subgroup of G and let f be an epimorphism of G onto a group G'. Then $f(A)$ is a fuzzy subgroup of G.*

Proof. Assume that A is a fuzzy subgroup of G and let $f(x) = u$ and $f(y) = v \in G'$. We can thus have $f\left(y^{-1}\right) = v^{-1}$. Since $A\left(xy^{-1}\right) \geq A(x) \wedge A(y)$, we have $f(A)\left(v^{-1}\right) \wedge f(A)(u) = \bigvee_{f(x)=u} A(x) \wedge \bigvee_{f(y^{-1})=v^{-1}} A\left(y^{-1}\right)$

$= \bigvee_{f(x)=u, f(y^{-1})=v^{-1}} \left(A(x) \wedge A\left(y^{-1}\right)\right)$

$\leq \bigvee_{f(x)=u, f(y^{-1})=v^{-1}} A\left(xy^{-1}\right)$

Since $f\left(y^{-1}\right)=v^{-1}$ and $f(x)=u$, we have $f\left(xy^{-1}\right)=uv^{-1}$

Thus,

$\bigvee_{f(x)=u, f(y^{-1})=v^{-1}} A\left(xy^{-1}\right)$

$= \bigvee_{f(xy^{-1})=uv^{-1}} A\left(xy^{-1}\right)$

$= f(A)\left(uv^{-1}\right)$

That is, $f(A)\left(v^{-1}\right) \wedge f(A)(u) \leq f(A)\left(uv^{-1}\right)$ ■

Proposition 310 *Let f be a homomorphism from G to a group $G\prime$ and let B be a fuzzy subgroup of G'. Then $f^{-1}(B)$ is a fuzzy subgroup of G.*

Proof. For any $x, y \in G$, $f^{-1}(B)(xy^{-1})$

$= B(f(xy^{-1})) = B(f(x)f(y^{-1}))$

$\geq B(f(x)) \wedge B(f(y^{-1}))$

$= B(f(x)) \wedge B(f(y)^{-1})$

$\geq B(f(x)) \wedge B(f(y))$

$= f^{-1}(B)(x) \wedge f^{-1}(B)(y)$. ■

Let $G_1, G_2, ..., G_n$ be n groups. We know from abstract algebra that $G_1 \times G_2 \times ... \times G_n$ is still a group under the multiplication defined $\forall x_i, y_i \in G_i$ for $i = 1, 2, ..., n$, $(x_1, x_2, ..., x_n) * (y_1, y_2, ..., y_n) = (x_1 * y_1, x_2 * y_2, ..., x_n * y_n)$. In this group, $(x_1, x_2, ..., x_n)^{-1} = (x_1^{-1}, x_2^{-1}, ..., x_n^{-1})$. In a similar vain, we have the following:

Proposition 311 *Let $A_1, A_2, ..., A_n$ be fuzzy subgroups of $G_1, G_2, ..., G_n$ respectively. Then the Cartesian product $A_1 \times A_2 \times ... \times A_n$ is a fuzzy subgroup of $G_1 \times G_2 \times ... \times G_n$.*

Proof. Form a tuple for $x_i, y_i \in G_i$. Since we have $A_i\left(x_i y_i^{-1}\right) \geq A_i(x_i) \wedge A_y(y_i)$

then, $(A_1 \times A_2 \times ... \times A_n)\left(x_1 y_1^{-1}, x_2 y_2^{-1}, ..., x_n y_n^{-1}\right)$

$= \bigwedge_{i=1}^{n} A_i\left(x_i y_i^{-1}\right)$

$\geq \bigwedge_{i=1}^{n} A_i(x_i) \wedge A_i(y_i)$

$(A_1 \times A_2 \times ... \times A_n)(x_1 y_1, x_2 y_2, ..., x_n y_n)$ ■

Definition 312 *A fuzzy subgroup A of G is called **normal** if $A(xy) = A(yx)$ holds for any $x, y \in G$*

Proposition 313 *A fuzzy subgroup A of G is normal iff $A(xyx^{-1}) = A(y)$ holds for any $x, y \in G$.*

Proof. Suppose A is normal. By definition, for any $x, y \in G$, $A(xyx^{-1}) = A(xx^{-1}y) = A(y)$.

Conversely, suppose $A(xyx^{-1}) = A(y)$ holds for any $x, y \in G$.Then $A(xy) = A(xyxx^{-1}) = A(yx)$, i.e. A is normal. ■

Proposition 314 *$A \in F(G)$ is a normal fuzzy subgroup of G iff $A \circ A^{-1} = A$ and $A \circ B = B \circ A$ holds for all $B \in F(G)$*

Proof. For any fuzzy subgroup, $A \circ A^{-1} = A$. Take $(A \circ B)(z)$

$$= \bigvee_{z=xy} A(x) \wedge B(y)$$
$$= \bigvee_{y \in G} A\left(zy^{-1}\right) \wedge B(y)$$
$$= \bigvee_{y \in G} A\left(y^{-1}z\right) \wedge B(y)$$
$$= \bigvee_{y \in G} A\left(y^{-1}z\right) \wedge B(y)$$
$$= \bigvee_{z=yx} A(x) \wedge B(y)$$
$$= \bigvee_{z=yx} B(y) \wedge A(x)$$
$$= (B \circ A)(z)$$

Conversely, $A \circ A^{-1} = A$ implies A is a fuzzy subgroup. To show that A is normal, take $B = \left\{x^{-1}\right\}$

Then, $A(xy) = \left(\left\{x^{-1}\right\} \circ A\right)(y) = \left(A \circ \left\{x^{-1}\right\}\right)(y) = \bigvee_{y=st} A(s) \wedge \left\{x^{-1}\right\}(t) = A(yx)$ ■

Proposition 315 *$A \in F(G)$ is a normal fuzzy subgroup of G iff A_α is a normal subgroup of G for any $\alpha \in \mathcal{R}(A)$*

Proof. A is a subgroup iff A_α is one. For normality, take $x \in G$ and $y \in A_\alpha$. It follows from $A(xyx^{-1}) = A(y) \geq \alpha$. Hence $xyx^{-1} \in A_\alpha$, and thus A_α is normal. Conversely, take $x, y \in G$ and $\alpha = A(y)$. Then $\alpha \in \{A(x) \mid x \in G\}$ and $y \in A_\alpha$. Hence $xyx^{-1} \in A_\alpha$.Consequently, $A(xy^{-1}x) \geq \alpha = A(y)$. As a result, A is a normal fuzzy subgroup of G. ■

Particularly, A^* is a normal subgroup of G if A is a normal fuzzy subgroup of G.

Definition 316 *Let A be a fuzzy subgroup of G. For every $x \in G$, define $xA, Ax \in F(G)$ by: $\forall y \in G$, $(xA)(y) = A(x^{-1}y)$ and $(Ax)(y) = A(yx^{-1})$.*

Then xA and Ax are called the left coset and right coset of A w.r.t. x respectively. Clearly, $xA = Ax$ holds for any $x \in G$ if A is a normal fuzzy subgroup of G. In this case, we simply call $xA(= Ax)$ a coset. Write $G/A = \{xA \mid x \in G\}$.

Lemma 317 *Let A be two normal fuzzy subgroups of G. Then $xA \circ yA = (xy)A$ holds for any two cosets $xA, yA \in G/A$*

Proof. On the one hand, for any $z \in G$, $(xA \circ yA)(z) = \bigvee_{z=z_1z_2} ((xA)(z_1) \wedge (yA)(z_2))$

$\geq (xA)(x) \wedge (yA)(x^{-1}z) = A(x^{-1}x) \wedge A(y^{-1}x^{-1}z) = A(e) \wedge A(y^{-1}x^{-1}z) = A((xy)^{-1}z) = ((xy)A)(z)$.

On the other hand, considering that A is normal,

$(xA \circ yA)(z) = \bigvee_{z=z_1z_2} ((xA)(z_1) \wedge (yA)(z_2))$

$= \bigvee_{z=z_1z_2} (A(x^{-1}z_1) \wedge A(y^{-1}z_2))$

$= \bigvee_{z=z_1z_2} (A(x^{-1}z_1) \wedge A(z_2y^{-1}))$

$\leq \bigvee_{z=z_1z_2} A(x^{-1}z_1z_2y^{-1})$

$= A(x^{-1}zy^{-1}) = A(y^{-1}x^{-1}z)$

$= A((xy)^{-1}z) = ((xy)A)(z)$. ■

We have the following result concerning $(G/A, \circ)$.

Proposition 318 *Let A be a normal fuzzy subgroup of G. Then*

(1) $(G/A, \circ)$ is a group and

(2) G/A is isomorphic to G/A.*

Proof. (1) Clearly, the operation $\circ$ is associative, A is the identity of G/A and the inverse of xA is $x^{-1}A$. Hence $(G/A, \circ)$ is a group.

(2) For any $x \in G$, let $f : xA \longrightarrow xA^*$. Then, for any $x, y \in G$,

$f(xA \circ yA) = f(xyA) = xyA^* = xA^*yA^* = f(xA)f(yA)$.

Hence f is a homomorphism. In order to prove that f is injective, suppose that $xA = yA$. Then $A(x^{-1}z) = A(y^{-1}z)$ for all $z \in G$. Particularly, $A(x^{-1}y) = A(e)$ when $z = y$. Thus $x^{-1}y \in A^*$. As a result, $xA^* = yA^*$. Hence f is injective. It is clear that f is surjective. In summary, f is an isomorphism between G/A and G/A^*. ■

G/A will be called the quotient group of G by a normal fuzzy subgroup A of G.

Proposition 319 *Let A be a normal fuzzy subgroup of G. Define $\bar{A} : G/A \longrightarrow [0,1]$ by: $\forall xA \in G/A, \bar{A}(xA) = A(x)$.*

Then $\bar{A}$ is a normal fuzzy subgroup of G/A.

Proof. Firstly, for any $xA \in G/A$, $\bar{A}((xA)^{-1}) = \bar{A}(x^{-1}A) = A(x^{-1}) = A(x) = \bar{A}(xA)$ and for any $xA, yA \in G/A$, $\bar{A}(xA \blacksquare yA) = \bar{A}(xyA) = A(xy) \geq A(x) \wedge A(y) = \bar{A}(xA) \wedge \bar{A}(yA)$. Hence $\bar{A}$ is a fuzzy subgroup of G/A. Next, for any xA, $yA \in G/A$, $\bar{A}(xA \circ yA) = \bar{A}(xyA) = A(xy) = A(yx) = \bar{A}(yxA) = \bar{A}(yA \circ xA)$. Hence $\bar{A}$ is a normal fuzzy subgroup of G/A. ■

Proposition 320 *Let A be a normal fuzzy subgroup of G and let f be an epimorphism of G onto a group G. Then $f(A)$ is a normal fuzzy subgroup of G.*

Proof. By Proposition 5.8, $f(A)$ is a fuzzy subgroup of G. Let $u, v \in G$. Then there exists $x \in G$ such that $f(x) = u$ since f is surjective. Hence, we obtain successively $f(A)(uvu^{-1}) = \bigvee_{f(z)=uvu^{=1}} A(z)$

$= \bigvee_{f(z)=f(x)v(f(x))^{=1}} A(z)$

$= \bigvee_{f(x^{-1}zx)=v} A(z)$ (f is a homomorphism)

$= \bigvee_{f(y)=v} A(xyx^{-1}) = \bigvee_{f(y)=v} A(y)$ (A is normal)

$= f(A)(v)$.

Hence $f(A)$ is a normal fuzzy subgroup of G. ∎

Proposition 321 *Let f be a homomorphism from G to a group G and let B be a normal fuzzy subgroup of G. Then $f^{-1}(B)$ is a normal fuzzy subgroup of G.*

Proof. By Proposition 5.9, $f^{-1}(B)$ is a fuzzy subgroup of G. Now, let $x, y \in G$. Then $f^{-1}(B)(xy) = B(f(xy)) = B(f(x)f(y)) = B(f(y)f(x)) = B(f(yx)) = f^{-1}(B)(yx)$.

Hence $f^{-1}(B)$ is a normal fuzzy subgroup of G. ∎

Proposition 322 *Let $A_1, A_2, ..., A_n$ be normal fuzzy subgroups of $G_1, G_2, ..., G_n$ respectively. Then the Cartesian product $\prod_{i=1}^{n} A_i$ is a normal fuzzy subgroup of $G_1 \times G_2 \times ... \times G_n$.*

Proof. By Proposition 5.10, $\prod_{i=1}^{n} A_i$ is a fuzzy subgroup of $G_1 \times G_2 \times ... \times G_n$

Furthermore, $\forall (x_1, x_2, \ldots, x_n), (y_1, y_2, ..., y_n) \in G_1 \times G_2 \times ... \times G_n$

$\left(\prod_{i=1}^{n} A_i\right)(x_1, x_2, \ldots, x_n), (y_1, y_2, ..., y_n)$

$= \left(\prod_{i=1}^{n} A_i\right)(x_1y_1, x_2y_2, \ldots, x_ny_n)$

$= \bigwedge_{i=1}^{n} A_i\,(x_iy_i)$

$= \bigwedge_{i=1}^{n} A_i\,(y_ix_i)$

$= \left(\prod_{i=1}^{n} A_i\right)(y_1x_1, y_2x_2, \ldots, y_nx_n)$ ∎

3.2.2 Fuzzy Subrings

In this and next subsection, we assume $(R, +, \circ)$ is a ring. For convenience, we write xy instead of $x \circ y$ for $x, y \in R$.

Definition 323 *$A \in F(R)$ is called a fuzzy subring of R if A satisfies that*
(1)$\forall x, y \in R, A(x - y) \geq A(x) \wedge A(y)$ and
(2) $\forall x, y \in R, A(xy) \geq A(x) \wedge A(y)$.

From the definition, it follows that A is a fuzzy subgroup of R under addition $+$ if A is a fuzzy subring of R. Furthermore, this fuzzy subgroup is normal since the addition is commutative. As a result, $\forall x \in R, A(x) \leq A(0)$ for every fuzzy subring A, where 0 denotes the zero element of R.

Proposition 324 *$A \in F(R)$ is a fuzzy subring of R iff A_α is a subring of R for every $\alpha \in \mathcal{R}(A)$.*

Proof. The proof is similar to that of Proposition 5.6. ■

By Proposition 5.19, $A^*= \{x \mid A(x) = A(0)\}$ is a subring of R. The operations on R can be extended to $F(R)$ as follows: $\forall A, B \in F(R), \forall z \in R$,

$(A+B)(z) = \bigvee_{x+y=z} (A(x) \wedge B(y))$;

$(A-B)(z) = \bigvee_{x-y=z} (A(x) \wedge B(y))$;

$(A \circ B)(z) = \bigvee_{xy=z} (A(x) \wedge B(y))$

Remark 325 *$A \in F(R)$ is a fuzzy subring of R iff $A - A \subseteq A$ and $A \circ A \subseteq A$.*

Proof. Let A be a fuzzy subring of R. Since A is a fuzzy group under addition, $A - A \subseteq A$ by Proposition 5.7. Moreover, $\forall z \in R$, $(A \circ A)(z) = \bigvee_{xy=z} (A(x) \wedge A(y)) \leq A(xy) = A(z)$, i.e. $A \circ A \subseteq A$.

Conversely, suppose that $A - A \subseteq A$ and $A \circ A \subseteq A$. Then, $\forall x, y \in R$, $A(x-y) \geq (A-A)(x-y)$

$= \bigvee_{s-t=x-y} (A(s) \wedge A(t)) \geq A(x) \wedge A(y)$.

Similarly, $A(xy) \geq (A \circ A)(xy) = \bigvee_{xy=st} (A(s) \wedge A(t)) \geq A(x) \wedge A(y)$. Consequently, A is a fuzzy subring of R. ■

Proposition 326 *Let A be a fuzzy subring of R and let f be an epimorphism of R onto a ring R. Then $f(A)$ is a fuzzy subring of R.*

Proof. Let $u, v \in R$. Then there exist $x, y \in R$ such that $f(x) = u$ and $f(y) = v$ since f is surjective. Hence, we obtain successively $f(A)(u) \wedge f(A)(v) =$

$\left(\bigvee_{f(x)=u} A(x) \right) \wedge \left(\bigvee_{f(y)=v} A(y) \right)$

$= \bigvee_{f(x)=u, f(y)=v} A(x) \wedge A(y)$

$\leq \bigvee_{f(x)=u, f(y)=v} A(x-y)$ (A is a fuzzy subring of R)

$\leq \bigvee_{f(x)-f(y)=u-v} A(x-y)$ (f is a homomorphism)

$= \bigvee_{f(z)=u-v} A(z) = f(A)(u-v)$.

Similarly,

$f(A)(uv) \geq f(A)(u) \wedge f(A)(v)$.

Hence, $f(A)$ is a fuzzy subring of R. ■

Proposition 327 *Let f be a homomorphism from R to a ring R and let B be a fuzzy subring of R. Then $f^{-1}(B)$ is a fuzzy subring of R.*

Proof. For any $x, y \in R$, $f^{-1}(B)(xy) = B(f(xy)) = B(f(x)f(y)) \geq B(f(x)) \wedge B(f(y)) = f^{-1}(B)(x) \wedge f^{-1}(B)(y)$. Similarly, $f^{-1}(B)(x-y) \geq f - 1(B)(x) \wedge f^{-1}(B)(y)$. Thus $f^{-1}(B)$ is a fuzzy subring of R. ■

Definition 328 *A fuzzy subring A of R is called a fuzzy ideal of R if it satisfies that, for any $x, y \in R$, $A(xy) \geq A(x) \vee A(y)$.*

Clearly, $A \in F(R)$ is a fuzzy ideal of R iff A satisfies that, $\forall x, y \in R$, $A(x-y) \geq A(x) \wedge A(y)$ and $A(xy) \geq A(x) \vee A(y)$. If R is commutative, then a fuzzy subring A of R is a fuzzy ideal iff R satisfies that, for any $x, y \in R$, $A(xy) \geq A(x)$.

Proposition 329 *Let $A \in F(R)$. Then A is a fuzzy ideal of R iff A_α is an ideal of R for every $\alpha \in \mathcal{R}(A)$.*

Proof. Firstly, suppose that A is a fuzzy ideal of R. Then, A_α for $\alpha \in \mathcal{R}(A)$ is a subring of R. Let $x, y \in A_\alpha$ and $z \in R$. Then $A(x-y) \geq A(x) \wedge A(y) \geq \alpha$ and $A(zx) \geq A(z) \vee A(x) \geq A(x) \geq \alpha$. Hence, $x - y \in A_\alpha$ and $zx \in A_\alpha$. Thus A_α is an ideal of R.

Conversely, suppose that Aα is an ideal of R for every $\alpha \in \mathcal{R}(A)$. Then, A is a fuzzy subring of R. Let $x, y \in R$ and $\alpha = A(x)$. Then $\alpha \in \mathcal{R}(A)$ and $x \in A_\alpha$. Since A_α is an ideal, $xy \in A_\alpha$. Hence $A(xy) \geq \alpha = A(x)$. Similarly, $A(xy) \geq A(y)$. Therefore, $A(xy) \geq A(x) \vee A(y)$. Thus A is a fuzzy ideal of R. Particularly, $A^* = \{x \mid A(x) = A(0)\}$ is an ideal of R if A is a fuzzy ideal of R. ■

Proposition 330 *Let A be a fuzzy ideal of R and let f be an epimorphism of R onto a ring R . Then $f(A)$ is a fuzzy ideal of R.*